MATHEMATICS OF DEGREE
PART ONE
THE MATHEMATICS OF DEGREE, CATEGORY $\overset{1}{C}$
REAL NUMBERS OF DEGREE

1. FUNDAMENTAL CONCEPTS AND DEFINITIONS

Definition: The notation,

$$\overset{\kappa}{\alpha} \qquad (1.1)$$

represents a number of degree.
Where α and κ are real numbers. α shall be called the base and κ the hyperthetis (degree) of the number of degree (1.1). The number κ signifies the degree of the number α.

Definition: The mathematical operations, applied to the hyperthetis and the base of a number of degree, are the equivalent familiar operations of real numbers.

Definition: The mathematics of degree are founded in two categories:

a. **In the axiomatic foundation of category $\overset{1}{C}$, and**
b. **In the axiomatic foundation of category $\overset{0}{C}$.**

Definition: According to the mathematics of degree, the hitherto known real numbers are considered numbers of degree, i.e.:

a. **Degree κ = 1, of the axiomatic foundation $\overset{1}{C}$, or**
b. **Degree κ = 0, of the axiomatic foundation $\overset{0}{C}$.**

Definition: The degree,

a. **κ = 1, of the axiomatic foundation $\overset{1}{C}$, and**
b. **κ = 0, of the axiomatic foundation $\overset{0}{C}$,**

will be called fundamental degrees of the mathematics of degree.

Based on the definitions stated above, it is immediately obvious that the hitherto known real numbers are a partial case of the numbers of degree.

This means that:
If, to any mathematical formula that arises from the axiomatic foundation $\overset{1}{C}$ or $\overset{0}{C}$, we assign,

a. **κ = 1, (when referring to category $\overset{1}{C}$), and**
b. **κ = 0, (when referring to category $\overset{0}{C}$),**

then, from the above mentioned mathematical formulae of the mathematics of degree, the equivalent mathematical formulae of the hitherto known mathematics necessarily result.

NOTE: When referring to the axiomatic foundation of category $\overset{1}{\mathbf{C}}$, κ = 1 is implied for numbers not bearing a hyperthetis, it is simply omitted for simplicity.
Similarly, when referring to the axiomatic foundation of category $\overset{0}{\mathbf{C}}$, κ = 0 is implied for numbers not bearing a hyperthetis, again omitted for simplicity.

2. AXIOMATIC FOUNDATION OF REAL NUMBERS OF DEGREE, CATEGORY $\overset{1}{\mathbf{C}}$

BASIC OPERATIONS ON NUMBERS OF DEGREE, CATEGORY $\overset{1}{\mathbf{C}}$

1. Addition

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 = \overset{\kappa\cdot\lambda}{(a_1 + a_2)}$$

($\boldsymbol{\alpha_1 + \alpha_2}$) is the base and **κ·λ** the hyperthetis.

2. Subtraction

$$\overset{\kappa}{a}_1 - \overset{\lambda}{a}_2 = \overset{\kappa\cdot\lambda}{(a_1 - a_2)}$$

($\boldsymbol{\alpha_1 - \alpha_2}$) is the base and **κ·λ** the hyperthetis.

3. Multiplication

$$\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2 = \overset{\kappa\cdot\lambda}{(a_1 \cdot a_2)}$$

($\boldsymbol{\alpha_1 \cdot \alpha_2}$) is the base and **κ·λ** the hyperthetis.

4. Division

$$\overset{\kappa}{a}_1 : \overset{\lambda}{a}_2 = \overset{\kappa\cdot\lambda}{(a_1 : a_2)}$$

($\boldsymbol{\alpha_1 : \alpha_2}$) is the base and **κ·λ** the hyperthetis.

Based on the above, it can be shown that:
If $\overset{\kappa}{a}_1$, $\overset{\lambda}{a}_2$, $\overset{\mu}{a}_3$ are elements of the set $\mathbf{S_g}$ of real numbers of degree, category $\overset{1}{\mathbf{C}}$, then:

a. $\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2$ and $\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2$ belong to $\mathbf{S_g}$ (closure property).

b. $\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 = \overset{\lambda}{a}_2 + \overset{\kappa}{a}_1$ (commutative property).

c. $\overset{\kappa}{a}_1 + (\overset{\lambda}{a}_2 + \overset{\mu}{a}_3) = (\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2) + \overset{\mu}{a}_3$ (associative property).

d. $\overset{\kappa}{a}_1 \cdot (\overset{\lambda}{a}_2 \cdot \overset{\mu}{a}_3) = (\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2) \cdot \overset{\mu}{a}_3$ (associative property).

e. $\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2 = \overset{\lambda}{a}_2 \cdot \overset{\kappa}{a}_1$ (commutative property).

f. $\overset{\kappa}{a}_1 \cdot (\overset{\lambda}{a}_2 + \overset{\mu}{a}_3) = \overset{\kappa}{a}_1 \cdot (a_2 \overset{\lambda\cdot\mu}{+} a_3)$ and not $\overset{\kappa}{a}_1 \cdot (\overset{\lambda}{a}_2 + \overset{\mu}{a}_3) = \overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2 + \overset{\kappa}{a}_1 \cdot \overset{\mu}{a}_3$ (non-distributive property).

g. $\overset{\kappa}{a}_1 + \overset{1}{0} = \overset{1}{0} + \overset{\kappa}{a}_1 = \overset{\kappa}{a}_1$.

$\overset{1}{0}$ is called the identity element of addition.

h. $\overset{\kappa}{a}_1 \cdot \overset{1}{1} = \overset{1}{1} \cdot \overset{\kappa}{a}_1 = \overset{\kappa}{a}_1$.

$\overset{1}{1}$ is called the identity element of multiplication.

i. For every real number of degree $\overset{\kappa}{a}_1$ there is only one real number of degree $\overset{\rho}{a}$ of $\mathbf{S_g}$, such that:

$$\overset{\kappa}{a}_1 + \overset{\rho}{a} = \overset{1}{0}.$$

$\overset{\rho}{a}$ is called the additive inverse of $\overset{\kappa}{a}_1$ and is denoted by:

$$-\overset{1/\kappa}{a_1}, \text{ where } \rho = \frac{1}{\kappa}.$$

j. For every $\overset{\kappa}{a}_1 \cdot \overset{\rho}{a} = \overset{\rho}{a} \cdot \overset{\kappa}{a}_1 = \overset{1}{1}$.

$\overset{\rho}{a}$ is called the multiplicative inverse of $\overset{\kappa}{a}_1$ and is denoted by:

$$\overset{1/\kappa}{\left(\frac{1}{a_1}\right)}$$

k. If:

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 = \overset{\mu}{a}_3 + \overset{\nu}{a}_4$$

then:

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 - \overset{1/\mu}{a}_3 = \overset{\nu}{a}_4$$

$$\overset{\kappa}{a}_1 = \overset{\mu}{a}_3 + \overset{\nu}{a}_4 - \overset{1/\lambda}{a}_2$$

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 - \overset{1/\mu}{a}_3 - \overset{1/\nu}{a}_4 = \overset{1}{0} \quad \text{(statement } \mathbf{o}\text{)}$$

From the above identity we notice that, in an equality with real numbers of degree in category $\overset{1}{C}$, when a term is moved from one side of the equation to the other, then, the sign of the base changes and the hyperthetis takes the value of its inverse.

IDENTITIES AND NUMBERS

Identity 1: If $\overset{\kappa}{a}$ is a real number of degree, then we have:

$$\overset{\kappa}{a} = \overset{1}{\alpha} \cdot \overset{\kappa}{1} = \alpha \cdot \overset{\kappa}{1}$$

Identity 2: According to the known, we have:

$$\sqrt{\overset{\kappa}{a}} = \overset{\sqrt{\kappa}}{\sqrt{a}}$$

Indeed:

$$\overset{\sqrt{\kappa}}{\sqrt{a}} \cdot \overset{\sqrt{\kappa}}{\sqrt{a}} = \left(\overset{\sqrt{\kappa}}{\sqrt{a}} \cdot \overset{\sqrt{\kappa}}{\sqrt{a}} \right)$$

$= \overset{\sqrt{\kappa^2}}{\sqrt{a^2}} = \overset{\kappa}{a}$, that is to say, the subroot of $\sqrt{\overset{\kappa}{a}}$, QED.

And in general:

$$\sqrt[\nu]{\overset{\kappa}{a}} = \overset{\sqrt[\nu]{\kappa}}{\sqrt[\nu]{a}}$$

Indeed:

$$\underbrace{\overset{\sqrt[\nu]{\kappa}}{\sqrt[\nu]{a}} \cdot \overset{\sqrt[\nu]{\kappa}}{\sqrt[\nu]{a}} \cdots \overset{\sqrt[\nu]{\kappa}}{\sqrt[\nu]{a}}}_{\nu - times} = \overset{\sqrt[\nu]{\kappa^\nu}}{\sqrt[\nu]{a^\nu}} = \overset{\kappa}{a} \text{ QED.}$$

Identity 3: Always:

$$\left(\overset{\kappa}{a} \right)^\nu = \overset{\kappa^\nu}{a^\nu}$$

Indeed:

$$\underbrace{\overset{\kappa}{a} \cdot \overset{\kappa}{a} \cdots \overset{\kappa}{a}}_{\nu - times} = \left(\overset{\kappa \cdot \kappa \cdot \ldots \cdot \kappa}{a \cdot a \cdots a} \right) = \overset{\kappa^\nu}{a^\nu}$$

Definition: We define as a power,

$$(a)^{\overset{\mu}{\nu}}$$

where **α** is the base and $\overset{\mu}{\nu}$ the index (itself a number of degree), the relationship:

$$(a)^{\overset{\mu}{\nu}} = \underbrace{\overset{\mu}{a} \cdot \overset{\mu}{a} \cdots \overset{\mu}{a}}_{\nu - times} = \overset{\mu^\nu}{a^\nu}$$

where a^ν is the base and μ^ν its hyperthetis.

Identity 4: The previous definition and identity 3 can be united to give:

Definition: By definition, we accept that the general form will be:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} = \underbrace{\overset{\kappa\cdot\mu}{a} \cdot \overset{\kappa\cdot\mu}{a} \cdot \overset{\kappa\cdot\mu}{a} \cdots \overset{\kappa\cdot\mu}{a}}_{\nu-times}$$

which reduces to:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} = \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}}$$

where, a^{ν} is the base and $(\kappa \cdot \mu)^{\nu}$ the hyperthetis.

E.g.: for **μ = 1**, we have:

$$\left(\overset{\kappa}{a}\right)^{\overset{1}{\nu}} = \left(\overset{\kappa}{a}\right)^{\nu} = \overset{\kappa^{\nu}}{a^{\nu}} \text{ (identity 3).}$$

Also, for **κ = 1** and **μ = 1**, we have:

$$\left(\overset{1}{a}\right)^{\overset{1}{\nu}} = a^{\nu},$$

etc..

Identity 5: For two numbers of degree $\overset{\kappa}{a}$ and $\overset{\kappa}{a}$, the following always holds:

$$\overset{\kappa}{a} + \overset{\kappa}{a} = \overset{\kappa\cdot\kappa}{(a+a)} = \left(\overset{\kappa^2}{2a}\right)$$

or

$$\overset{\kappa}{a} + \overset{\kappa}{a} = 2 \cdot \overset{\kappa}{a} \text{ (Only when } \boldsymbol{\kappa = 0} \text{ or } \boldsymbol{\kappa = 1}\text{)}$$

Identity 6: If:

$$a = a_1 + a_2 + a_3 + \ldots + a_{\nu}$$

then by definition:

$$\overset{\kappa}{a} = (a_1 + a_2 + \overset{\kappa}{a_3} + \ldots + a_{\nu})$$

Definition: Two real numbers of degree $\overset{\kappa}{a}$ and $\overset{\lambda}{\beta}$ are equal,

$$\overset{\kappa}{a} = \overset{\lambda}{\beta}$$

if and only if:

$$\left.\begin{array}{r} a=\beta \\ \text{and } \kappa=\lambda \end{array}\right\}$$

Identity 7: In an equality made up of two equal numbers of degree, if we add, subtract, multiply or divide both sides by the same number of degree, the equality remains unchanged.

EXAMPLES

For better understanding and practice in the mathematics of degree, it is judged necessary to mention a few representative examples of what has been stated above.

1. Calculate the sum:

$$A=\overset{3}{2}+\overset{5}{3}+\overset{-6}{2}-\overset{1,5}{3}+\overset{8}{0}$$

As we know, we have:

$$A=\left(\overset{3\cdot5\cdot(-6)\cdot1,5\cdot8}{2+3+2-3+0}\right)=\overset{-1080}{4}$$

I.e., a number of degree, with base **4** and hyperthetis **-1080** and it is obviously of degree **-1080**.

2. Calculate the product:

$$A=\overset{2}{2}\cdot\overset{-2}{3}\cdot\overset{3}{4}\cdot\left(\overset{-2}{-5}\right)$$

As we know, we have:

$$A=\left(\overset{2\cdot(-2)\cdot3\cdot(-2)}{2\cdot3\cdot4\cdot(-5)}\right)=\overset{24}{-120}$$

3. Calculate the ratio:

$$A=\overset{6}{4}:\overset{-3}{2}$$

The required ratio is:

$$A=\left(\overset{6:(-3)}{4:2}\right)=\overset{-2}{2}$$

4. Because as we know:

$$\overset{3}{2}+\overset{5}{3}=\left(\overset{3\cdot5}{2+3}\right)=\overset{15}{5}$$

i.e.,

$$\overset{3}{2} + \overset{5}{3} = \overset{15}{5}$$

then:

$$\overset{3}{2} = \overset{15}{5} - \overset{1/5}{3} = \left(\overset{15\cdot 1/5}{5 - 3}\right) = \overset{3}{2} \text{ QED}$$

5. Calculate the square root:

$$A = \sqrt{\overset{9}{17}}$$

As we know, we have:

$$A = \sqrt{\overset{\sqrt{9}}{17}} = \sqrt{\overset{3}{17}}$$

6. Calculate the following:

$$A = \sqrt[5]{\overset{6}{49}}$$

As we know, we have:

$$A = \sqrt[5]{\overset{\sqrt[5]{6}}{49}}$$

7. Calculate the power:

$$A = \left(\overset{3}{4}\right)^{4}$$

As we know, it is:

$$A = \left(\overset{3^4}{4^4}\right) = \overset{81}{256}$$

8. Calculate the power:

$$A = (5)^{\overset{2}{3}}$$

As we know, we have:

$$A = \left(\overset{2^3}{5^3}\right) = \overset{8}{125}$$

9. Calculate the power:

$$A = \left(\overset{1}{2}\right)^{\overset{1}{3}}$$

As we know, we have:

$$A=\overset{(1\cdot1)^3}{\left(2^3\right)}=\overset{1^3}{8}=\overset{1}{8}=8$$

10. For which values of x and y does the following equality hold?

$$\overset{x}{\left(x+y\right)}=\overset{3x-y}{3}$$

SOLUTION

As we know, for the above equality to hold, we must have:

$$\left.\begin{aligned} x+y&=3 \\ \text{and}\quad x&=3x-y \end{aligned}\right\}$$

Solving these simultaneous equations, we get,

$$\left.\begin{aligned} x&=1 \\ y&=2 \end{aligned}\right\}$$

which are the required values.

Indeed, for **x = 1** and **y = 2** the given equation becomes:

$$\overset{1}{\left(1+2\right)}=\overset{3\cdot1-2}{3}$$

$$\text{or}\ \overset{1}{3}=\overset{1}{3}\ \text{QED.}$$

11. Calculate the value of the following:

$$A=\frac{\overset{3}{2}\left(\overset{1}{3}+\overset{-4}{2}\right)+\overset{2}{3}\cdot\overset{3}{4}-\sqrt{\overset{9}{4}}+\sqrt[3]{\overset{8}{27}}}{\left(\overset{2}{3}\right)^3\cdot\left(\overset{3}{2}\right)^3+\overset{5}{1}\cdot\overset{1/5}{1}}$$

SOLUTION

$$A=\frac{\overset{3}{2}\overset{1\cdot(-4)}{\left(3+2\right)}+\overset{2\cdot3}{\left(3\cdot4\right)}-\overset{\sqrt{9}}{\sqrt{4}}+\overset{\sqrt[3]{8}}{\sqrt[3]{27}}}{\overset{2^3}{\left(3^3\right)}\overset{3^3}{\left(2^3\right)}+\overset{5\cdot1/5}{\left(1\cdot1\right)}}=$$

$$=\frac{\overset{3}{2}\cdot\overset{-4}{5}+\overset{6}{12}-\overset{3}{2}+\overset{2}{3}}{\overset{8}{27}\cdot\overset{27}{8}+\overset{1}{1}}=$$

$$\frac{\overset{-12}{10}+\overset{6}{12}-\overset{3}{2}+\overset{2}{3}}{\overset{216}{216}+\overset{1}{1}}=\frac{\overset{-12\cdot 6\cdot 3\cdot 2}{\left(10+12-2+3\right)}}{\overset{216\cdot 1}{\left(216+1\right)}}=$$

$$\frac{\overset{-432}{23}}{\overset{216}{217}}=\overset{-432/216}{\left(\frac{23}{217}\right)}=\overset{-2}{\left(\frac{23}{217}\right)}\cong \overset{-2}{0,105}$$

12. Calculate the following:

$$\left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right)^2$$

As we know, we have:

$$\left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right)^2=\left[\overset{\kappa\cdot\lambda}{\left(a+\beta\right)}\right]^2=\overset{\kappa^2\cdot\lambda^2}{\left(a+\beta\right)^2}$$

13. Calculate the value of the following:

$$\left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right)^{\frac{\mu}{\nu}}$$

As we know, we have:

$$\left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right)^{\frac{\mu}{\nu}}=\left[\overset{\kappa\cdot\lambda}{\left(a+\beta\right)}\right]^{\frac{\mu}{\nu}}=\overset{\mu^\nu\cdot\kappa^\nu\cdot\lambda^\nu}{\left(a+\beta\right)^\nu}=\overset{(\mu\cdot\kappa\cdot\lambda)^\nu}{\left(a+\beta\right)^\nu}$$

where, as we can see, the base $(a+\beta)^\nu$ is Newton's binomial theorem and $\left(\mu\cdot\kappa\cdot\lambda\right)^\nu$ its hyperthetis.

NOTEWORTHY ALGEBRAIC FORMULAE OF CATEGORY $\overset{1}{C}$

1. As stated above:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}=\overset{(\kappa\cdot\mu)^\nu}{a^\nu}$$

from which:

$$\left(\overset{\kappa}{a}\right)^{\nu}=\overset{\kappa^\nu}{a^\nu}\ \text{and}$$

$$\left(a\right)^{\overset{\mu}{\nu}}=\overset{\mu^\nu}{a^\nu}$$

2. Prove that:

$$\left[\left(\overset{\kappa}{a}\right)^{\overset{\tau}{\rho}}\right]^{\overset{\mu}{\nu}} = \overset{(\kappa\tau)^{\rho\nu}\cdot\mu^{\nu}}{a^{\rho\nu}}$$

Proof: According to the equation in (1) above, we have:

$$\left(\overset{\kappa}{a}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa\cdot\tau)^{\rho}}{a^{\rho}}$$, therefore:

$$\left[\left(\overset{\kappa}{a}\right)^{\overset{\tau}{\rho}}\right]^{\overset{\mu}{\nu}} = \left(\overset{(\kappa\tau)^{\rho}}{a^{\rho}}\right)^{\overset{\mu}{\nu}} = \overset{\left((\kappa\tau)^{\rho}\cdot\mu\right)^{\nu}}{a^{\rho\nu}} = \overset{(\kappa\tau)^{\rho\nu}\cdot\mu^{\nu}}{a^{\rho\nu}}$$, QED

3. Prove that:

$$\sqrt[\overset{\mu}{\nu}]{\overset{\kappa}{a}} = \overset{\sqrt[\nu]{\frac{\kappa}{\mu^{\nu}}}}{\sqrt[\nu]{a}}$$

Proof: we set:

$$\sqrt[\overset{\mu}{\nu}]{\overset{\kappa}{a}} = \overset{B}{A}$$

so we have:

$$\overset{\kappa}{a} = \left(\overset{B}{A}\right)^{\overset{\mu}{\nu}}$$ or $$\overset{\kappa}{a} = \overset{(\mathrm{B}\mu)^{\nu}}{\mathrm{A}^{\nu}}$$ from which we get:

$$\left.\begin{array}{r}(\mathrm{B}\mu)^{\nu} = \kappa \\ \text{and } A^{\nu} = a\end{array}\right\}$$

which are rearranged to give:

$$\left.\begin{array}{l}B = \sqrt[\nu]{\dfrac{\kappa}{\mu^{\nu}}} \\ A = \sqrt[\nu]{a}\end{array}\right\} \text{ QED.}$$

4. Prove that:

$$\sqrt[\overset{\tau}{\rho}]{\sqrt[\overset{\mu}{\nu}]{\overset{\kappa}{a}}} = \overset{\sqrt[\rho\nu]{\frac{\kappa}{\mu^{\nu}\cdot\tau^{\rho\nu}}}}{\sqrt[\rho\nu]{a}}$$

Proof: According to the equation in (3) above, we have:

$$\sqrt[\substack{\mu \\ \nu}]{a}^{\kappa} = \sqrt[\nu]{a}^{\sqrt[\nu]{\frac{\kappa}{\mu^{\nu}}}} \text{, therefore:}$$

$$\sqrt[\substack{\tau \\ \rho}]{\sqrt[\substack{\mu \\ \nu}]{a}^{\kappa}} = \sqrt[\substack{\tau \\ \rho}]{\sqrt[\nu]{a}^{\sqrt[\nu]{\frac{\kappa}{\mu^{\nu}}}}} = \sqrt[\rho\nu]{a}^{\sqrt[\rho\nu]{\frac{\kappa}{\mu^{\nu}\cdot\tau^{\rho\nu}}}} \text{, QED.}$$

5. Prove that:

$$\sqrt[\substack{\mu \\ \nu}]{a}^{\kappa} \cdot \sqrt[\substack{\mu \\ \nu}]{\beta}^{\lambda} = \sqrt[\nu]{a\cdot\beta}^{\sqrt[\nu]{\frac{\kappa\cdot\lambda}{(\mu^{\nu})^{2}}}}$$

Proof: As we know, we have

$$\sqrt[\substack{\mu \\ \nu}]{a}^{\kappa} = \sqrt[\nu]{a}^{\sqrt[\nu]{\frac{\kappa}{\mu^{\nu}}}} \text{, and}$$

$$\sqrt[\substack{\mu \\ \nu}]{\beta}^{\lambda} = \sqrt[\nu]{\beta}^{\sqrt[\nu]{\frac{\lambda}{\mu^{\nu}}}}$$

Multiplying the parts we get:

$$\sqrt[\substack{\mu \\ \nu}]{a}^{\kappa} \cdot \sqrt[\substack{\mu \\ \nu}]{\beta}^{\lambda} = \sqrt[\nu]{a}^{\sqrt[\nu]{\frac{\kappa}{\mu^{\nu}}}} \cdot \sqrt[\nu]{\beta}^{\sqrt[\nu]{\frac{\lambda}{\mu^{\nu}}}} = \sqrt[\nu]{a\cdot\beta}^{\sqrt[\nu]{\frac{\kappa\cdot\lambda}{(\mu^{\nu})^{2}}}} \quad \text{QED}$$

6. Prove that:

$$\sqrt[\substack{\mu \\ \nu}]{a}^{\kappa} : \sqrt[\substack{\mu \\ \nu}]{\beta}^{\lambda} = \sqrt[\nu]{\frac{a}{\beta}}^{\sqrt[\nu]{\frac{\kappa}{\lambda}}}$$

Proof: Exactly as previous example

7. Prove that:

$$\sqrt[\substack{\mu \\ \nu}]{\left(\overset{\kappa}{a}\right)^{\frac{\tau}{\rho}}} = \sqrt[\nu]{a^{\rho}}^{\sqrt[\nu]{\frac{(\kappa\cdot\tau)^{\rho}}{\mu^{\nu}}}} = a^{\frac{\rho}{\nu}\left(\frac{(\kappa\tau)^{\rho}}{\mu^{\nu}}\right)^{\frac{1}{\nu}}}$$

Proof: As we know, we have

$$\binom{\kappa}{a}^{\overset{\tau}{\rho}} = \overset{(\kappa\tau)^{\rho}}{a^{\rho}}$$

Therefore, again from what we know:

$$\sqrt[\overset{\mu}{\nu}]{\binom{\kappa}{a}^{\overset{\tau}{\rho}}} = \sqrt[\overset{\mu}{\nu}]{\overset{(\kappa\tau)^{\rho}}{a^{\rho}}} = \sqrt[\nu]{\overset{\sqrt[\nu]{\frac{(\kappa\cdot\tau)^{\rho}}{\mu^{\nu}}}}{a^{\rho}}} = \overset{\left(\frac{(\kappa\tau)^{\rho}}{\mu^{\nu}}\right)^{\frac{1}{\nu}}}{a^{\frac{\rho}{\nu}}}$$, QED.

8. Prove that:

$$\underbrace{\binom{\kappa}{a}^{\overset{\mu}{\nu}} \cdot \binom{\kappa}{a}^{\overset{\mu}{\nu}} \cdot \binom{\kappa}{a}^{\overset{\mu}{\nu}} \;\mathbf{.....}\; \binom{\kappa}{a}^{\overset{\mu}{\nu}} = \binom{\kappa}{a}^{\overset{\mu}{\nu\cdot\rho}}}_{\rho-times}$$

Proof: As we know, the above can be written as:

$$\underbrace{\overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}} \cdot \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}} \cdot \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}} \cdots \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}}}_{\rho-times} = \overset{(\kappa\mu)^{\nu\cdot\rho}}{a^{\nu\rho}} = \binom{\kappa}{a}^{\overset{\mu}{\nu\rho}}$$, QED.

9. Prove that:

$$\underbrace{\binom{\kappa}{a}^{\overset{\mu}{\nu}} + \binom{\kappa}{a}^{\overset{\mu}{\nu}} + \binom{\kappa}{a}^{\overset{\mu}{\nu}} \;\mathbf{.....}\; \binom{\kappa}{a}^{\overset{\mu}{\nu}} = \rho\cdot \overset{(\kappa\cdot\mu)^{\nu\cdot\rho}}{a^{\nu}}}_{\rho-times}$$

Proof: As we know, the above can be written as:

$$\underbrace{\overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}} + \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}} + \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}} \cdots \overset{(\kappa\cdot\mu)^{\nu}}{a^{\nu}}}_{\rho-times} = \rho\overset{(\kappa\mu)^{\nu\cdot\rho}}{a^{\nu}}$$, QED.

10. It can also be shown that:

a) $$\binom{\kappa}{a}^{\overset{\mu}{\nu}} + \binom{\lambda}{\beta}^{\overset{\tau}{\rho}} = \overset{(\kappa\mu)^{\nu}\cdot(\lambda\tau)^{\rho}}{\left(a^{\nu}+\beta^{\rho}\right)}$$

b) $$\binom{\kappa}{a}^{\overset{\mu}{\nu}} - \binom{\lambda}{\beta}^{\overset{\tau}{\rho}} = \overset{(\kappa\mu)^{\nu}\cdot(\lambda\tau)^{\rho}}{\left(a^{\nu}-\beta^{\rho}\right)}$$

c) $$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} \cdot \left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa\mu)^{\nu}\cdot(\lambda\tau)^{\rho}}{\left(a^{\nu}\cdot\beta^{\rho}\right)}$$

d) $$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} : \left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa\mu)^{\nu}:(\lambda\tau)^{\rho}}{\left(a^{\nu}:\beta^{\rho}\right)}$$

The above proofs are very easy, based on what has been mentioned so far.

11. If: $\left(\overset{\kappa}{a}\right)^{\overset{y}{x}} = \overset{B_1}{A_1}$, then by definition:

$$\log_{\overset{\kappa}{a}} \overset{B_1}{A_1} = \overset{y}{x} \qquad \text{(B)}$$

where, $\overset{\kappa}{a}$ is the base of the logarithm $\overset{B_1}{A_1}$.

Because we have:

$$\left(\overset{\kappa}{a}\right)^{\overset{y}{x}} = \overset{(\kappa y)^x}{a^x}$$

then from these relationships we get:

$$\overset{(\kappa y)^x}{a^x} = \overset{B_1}{A_1} \qquad \text{or}$$

$$\left.\begin{aligned} (\kappa y)^x &= B_1 \\ \alpha^x &= A_1 \end{aligned}\right\} \text{(c)}$$

Solving the simultaneous equations **(c)** for **x** and **y** we find the value of the logarithm (base $\overset{\kappa}{a}$) of the number of degree $\overset{B_1}{A_1}$.

12. $$\overset{\kappa}{a}, \left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right), \left[\left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right)+\overset{\lambda}{\beta}\right], \ldots\ldots\ldots = \overset{\kappa}{a}, \overset{\kappa\cdot\lambda}{(a+\beta)}, \overset{\kappa\lambda^2}{(a+2\beta)}, \ldots\ldots\ldots\ldots \overset{\kappa\cdot\lambda^{\nu-1}}{(a+(\nu-1)\beta)}$$

is an arithmetic progression of degree (ν terms), with first term, $\overset{\kappa}{a}$, and common difference, $\overset{\lambda}{\beta}$.

13. For the above arithmetic progression, prove that:

a) The last term of the sequence is:

$$\tau = \overset{\kappa\cdot\lambda^{\nu-1}}{(a+(\nu-1)\beta)}$$

b) The sum of ν terms is:

$$\Sigma = \left(\overset{\kappa^{\nu}\cdot\lambda^{\left(\frac{\nu-1}{2}\right)^{\nu}}}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right)$$

Proof: a) By definition, the νth term of an arithmetic progression is:

$$\tau = \overset{\kappa\cdot\lambda^{\nu-1}}{(a+(\nu-1)\beta)}$$

b) To find the sum of ν terms we proceed as follows:

$$\Sigma = \overset{\kappa}{a} + \overset{\kappa\cdot\lambda}{(a+\beta)} + \overset{\kappa\lambda^{2}}{(a+2\beta)} + \dots\dots\dots\dots + \overset{\kappa\cdot\lambda^{\nu-1}}{(a+(\nu-1)\beta)}$$

$$\text{or } \Sigma = \left(\overset{\kappa\cdot\kappa\lambda\cdot\kappa\lambda^{2}\cdot\ldots\cdot\kappa\lambda^{\nu-1}}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right) = \left(\overset{\kappa^{\nu}\cdot\lambda^{1+2+3+\dots\dots\nu-1}}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right)$$

but:

$$\mathbf{1+2+3+\ldots+(\nu-1)} = \left(\frac{\nu-1}{2}\right)\nu$$

which gives us:

$$\Sigma = \left(\overset{\kappa^{\nu}\cdot\lambda^{\left(\frac{\nu-1}{2}\right)^{\nu}}}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right), \text{ QED}$$

14. $\overset{\kappa}{a}, \left(\overset{\kappa}{a}\cdot\overset{\lambda}{\beta}\right), \left[\left(\overset{\kappa}{a}\cdot\overset{\lambda}{\beta}\right)\cdot\overset{\lambda}{\beta}\right], \dots\dots\dots = \overset{\kappa}{a}, \overset{\kappa\cdot\lambda}{(a\cdot\beta)}, \overset{\kappa\lambda^{2}}{(a\cdot\beta^{2})}, \dots\dots\dots\dots \overset{\kappa\cdot\lambda^{\nu-1}}{(a\cdot\beta^{(\nu-1)})}$,

is a geometric progression of degree (ν terms), with first term, $\overset{\kappa}{a}$, and common ratio, $\overset{\lambda}{\beta}$.

a) The νth term of the sequence is:

$$\tau = \overset{\kappa\lambda^{\nu-1}}{(a\beta^{\nu-1})}$$

b) The sum of ν terms is:

$$\Sigma = \left[\overset{\kappa^{\nu}\cdot\lambda^{\left(\frac{\nu-1}{2}\right)^{\nu}}}{\frac{a(\beta^{\nu}-1)}{\beta-1}} \right]$$

These are proven in the manner demonstrated for the arithmetic progression.

15. Other sequences of degree:

$$\binom{\kappa}{a}^{\overset{\mu}{\nu}},\binom{\kappa}{a}^{\overset{\mu}{\nu}}+\binom{\lambda}{\beta}^{\overset{\tau}{\rho}},\binom{\kappa}{a}^{\overset{\mu}{\nu}}+\binom{\lambda}{\beta}^{\overset{\tau}{\rho}}+\binom{\lambda}{\beta}^{\overset{\tau}{\rho}},\ \ldots.$$

arithmetic progression with first term, $\binom{\kappa}{a}^{\overset{\mu}{\nu}}$, and common difference $\binom{\lambda}{\beta}^{\overset{\tau}{\rho}}$.

Also:

$$\binom{\kappa}{a}^{\overset{\mu}{\nu}},\binom{\kappa}{a}^{\overset{\mu}{\nu}}\cdot\binom{\lambda}{\beta}^{\overset{\tau}{\rho}},\binom{\kappa}{a}^{\overset{\mu}{\nu}}\cdot\binom{\lambda}{\beta}^{\overset{\tau}{\rho}}\cdot\binom{\lambda}{\beta}^{\overset{\tau}{\rho}}\ \ldots.$$

geometric progression with first term, $\binom{\kappa}{a}^{\overset{\mu}{\nu}}$, and common ratio $\binom{\lambda}{\beta}^{\overset{\tau}{\rho}}$.

NOTE: All the above mathematical formulae, can be reduced to the equivalent formulae of the hitherto known mathematics by setting the hyperthetes of the numbers of degree equal to 1.

GRAPHICAL REPRESENTATION OF NUMBERS OF DEGREE

The above mentioned numbers of degree can be represented graphically on a plane as follows:

Consider fig. 1, the plane E_g on which an orthogonal set of axes, x-y (consisting of real numbers), is marked.

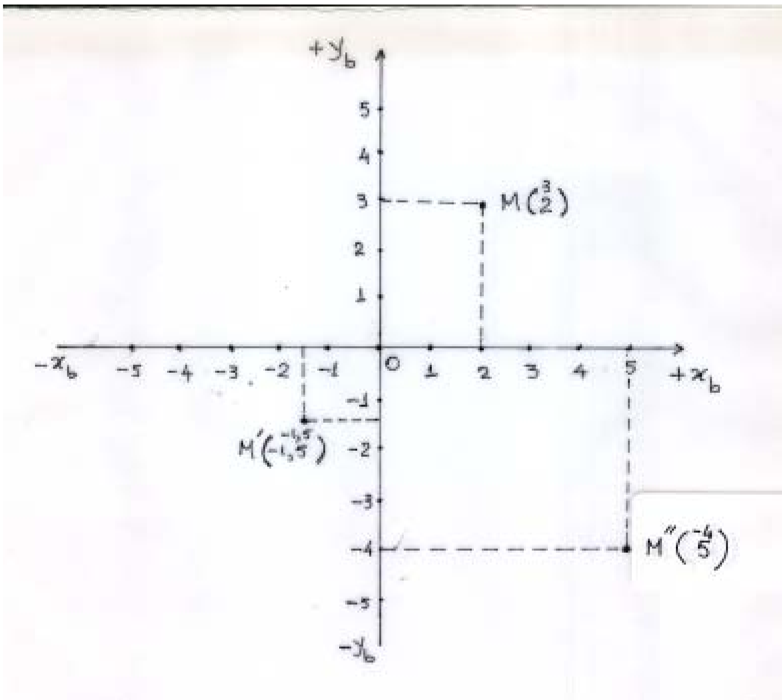


fig. 1

Now consider the number of degree $\overset{3}{2}$. On the x-axis we take represent the base (e.g. 2) and on the y-axis the hyperthetis (3). So, a point **M** on the x-y plane, with coordinates (2,3) corresponds to the number of degree $\overset{3}{2}$.

Similarly, the number of degree $\overset{-1.5}{-1.5}$ corresponds to the point (-1.5,-1.5) on the plane and in general any number of degree, $\overset{\kappa_i}{a_i}$, has a unique correspondence to the point $M_i = (a_i, \kappa_i)$.

Based on what has been stated above, the x-axis shall be called the base axis and the y-axis shall be called the hyperthetis axis of the real numbers of degree. The plane $\mathbf{E_g}$ shall be called the plane of real numbers of degree.

As we can see, on the plane $\mathbf{E_g}$ of the real numbers of degree, any point **M** corresponds to a unique number of degree and any number of degree corresponds to a unique point **M** on the plane.

Definition: We define the modulus, $|A|$, of a real number of degree, $\overset{\kappa}{a}$, as:

$$|A| = \sqrt{a^2 + \kappa^2}$$

for example, the modulus of the number of degree $\overset{3}{2}$ is:

$$|A| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

The modulus, $|A|$, is the length $\overleftrightarrow{OM}$.

3. FUNCTIONS OF DEGREE

The notation:

$$y_g = f_g(x) \qquad \textbf{(3.1)}$$

represents a function of degree of the real variable x, where $\mathbf{y_g}$ is the dependent variable.

In (3.1), because $\mathbf{y_g}$ and $\mathbf{f_g(x)}$ are numbers of degree, they will have the form:

$$\left. y_g = \overset{y_h}{y_b} \right\} \quad (3.2)$$

and $$\left. f_g(x) = \begin{matrix} f_2(x) \\ f_1(x) \end{matrix} \right\} \quad (3.3)$$

Based on (3.2) and (3.3), the equation of degree, (3.1), is written as:

$$\boxed{y_g = \frac{y_h}{y_b} = f_g(x) = \frac{f_2(x)}{f_1(x)}} \quad \textbf{(3.4)}$$

where:

The function:

$$y_h = f_2(x) \qquad (3.5)$$

is the hyperthetis function and,

$$y_b = f_1(x) \qquad (3.6)$$

is the base function, of the function of degree (3.1).

By definition, the function of degree (3.1), i.e.,

$$y_g = f_g(x) \qquad (3.7)$$

is equivalent to the functions (3.5) and (3.6), i.e.,

$$\left.\begin{array}{l} y_h = f_2(x) \\ y_b = f_1(x) \end{array}\right\} \qquad (3.8)$$

The functions (3.8), are the conjugated equations of the function of degree (3.1).

In an equation of degree,

$$y_g = \frac{f_2(x)}{f_1(x)} \qquad (3.9)$$

the following cases are possible:

a. $f_2(x) =$ **constant,** and
$f_1(x) =$ **function of x.**

b. $f_2(x) =$ **function of x,** and
$f_1(x) =$ **constant.**

c. $f_2(x) =$ **function of x,** and
$f_1(x) =$ **function of x.**

INQUIRY INTO THE FUNCTIONS OF DEGREE

A. Single variable functions of degree

Consider the function of degree:

$$y_g = 3^{2\;x+1} \cdot x + 1^{6} \qquad (3.10)$$

in the variable **x**.

Function (3.10) becomes:

$$y_g = 3^{2\;x+1} \cdot x + 1^{6} = \left(3 \cdot x\right)^{2(x+1)} + 1^{6} = \left(3x+1\right)^{2(x+1)6} \Leftrightarrow$$

$$y_g = \left(3x+1\right)^{12(x+1)} \qquad (3.11)$$

Consequently, from (3.11) we get the conjugate equations of the given function (3.10), as,

$$\left.\begin{aligned} y_h &= 12(x+1) \\ y_b &= 3x+1 \end{aligned}\right\} \qquad (3.12)$$

where, y_h is the conjugate hyperthetis function and y_b is the conjugate base function of the function of degree (3.10).

Consider now a tri-orthogonal system of coordinates (x,y_b,y_h).

Setting a value to the variable x, e.g. **x = 2**, then from (3.11) or (3.12), we have:

$$y_g = (3 \cdot 2 + 1)^{12(2+1)} = 7^{36}\text{, i.e.:}$$

$$y_h = 36 \text{ and } y_b = 7$$

Therefore, in the (x,y_b,y_h) coordinate system of figure 2, we have the point $\mathbf{M_1}$ with coordinates **(x = 2, y_b = 7, y_h = 36)**, i.e., the point:

$$M_1 = (2,7,36) \qquad (3.13)$$

Similarly, by setting a different value to the variable x, e.g. **x = 3**, then from (3.11) or (3.12), we have:

$$y_g = \left(3 \cdot 3 + 1\right)^{12(3+1)} = 10^{48}, \Rightarrow \text{, i.e.:}$$

$$y_h = 48 \text{ and } y_b = 10$$

Therefore, in the (x,y_b,y_h) coordinate system of figure 2, we have the point $\mathbf{M_2}$ with coordinates **($\mathbf{x = 3}$, $\mathbf{y_b = 10}$, $\mathbf{y_h = 48}$)**, i.e., the point:

$$M_2 = (3,10,48) \qquad (3.13)$$

and so forth, for any value $x_i, (i = 1,2,3,...)$ that can be assigned to the variable x, we get a point M_i with coordinates:

$$M_i = \left(x_i, y_{b,i}, y_{h,i}\right) \qquad \mathbf{(3.14)}$$

Obviously, the points M_i of (3.14), all lie on the curve C of the given function of degree (3.10).

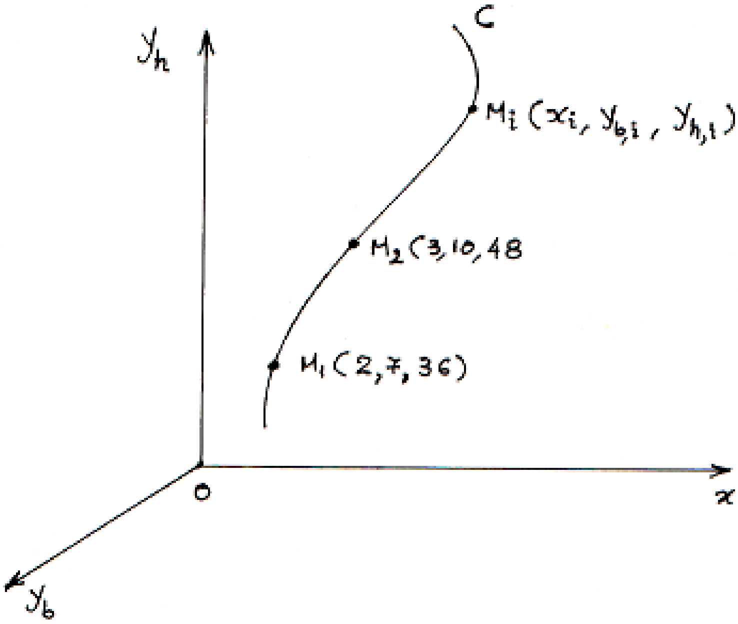


fig. 2

Conclusion:

A function of degree in one real variable, x, can be represented graphically on a 3-dimensional Euclidean space (tri-orthogonal coordinate system (x,y_b,y_h)**) as a curve C, figure 2.**

B. Two variable functions of degree

In the manner above, we can consider functions of degree in two variables **x** and **y**.

Suppose we have a function of degree:

$$z_g = x^{y} + y^{x} + 1^{2} \qquad (3.15)$$

with two variables **x, y**.

Function (3.15) can be rearranged to give:

$$z_g = \overset{y\cdot x}{(x+y)} + \overset{2}{1} = \overset{2xy}{\left(x+y+1\right)} \Leftrightarrow$$

$$z_g = \overset{2xy}{\left(x+y+1\right)} \qquad \textbf{(3.16)}$$

Consequently, the conjugate equations of the given function (3.15) are:

$$\left.\begin{aligned} y_h &= 2xy \\ y_b &= x+y+1 \end{aligned}\right\} \qquad \textbf{(3.17)}$$

Now consider figure 3, a 4-dimensional space $\mathbf{E^4}$ with an orthogonal system of coordinates (x,y,y_b,y_h).

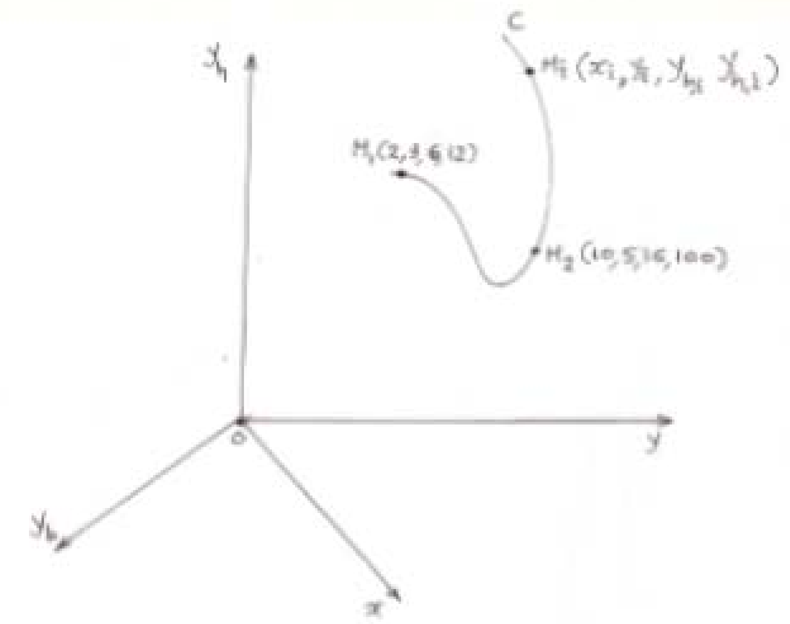


fig. 3

If we assign the values to the variables **x** and **y**, e.g. **x = 2** and **y = 3**, then from (3.16) or (3.17), we get:

$$Z_g = \overset{2\cdot 2\cdot 3}{\left(2+3+1\right)} = \overset{12}{6}\text{, i.e.,} \qquad (3.18)$$

$$y_h = 12 \ \text{ and } \ y_b = 6$$

Therefore, in the coordinate system (x,y,y_b,y_h) of figure 3, we have the point $\mathbf{M_1}$ with coordinates $(\mathbf{x = 2},\ \mathbf{y = 3},\ \mathbf{y_b = 6},\ \mathbf{y_h = 12})$, i.e., the point:

$$M_1 = (2,3,6,12) \qquad (3.19)$$

Similarly, by setting different values to the variables x and y, e.g. **x = 3**, then from (3.17) we have:

$$Z_g = \overset{2\cdot 10\cdot 5}{\left(10+5+1\right)} = \overset{100}{16}\text{, i.e.,} \qquad (3.20)$$

$$y_h = 100 \quad \text{and} \quad y_b = 16,$$

Therefore, in the coordinate system (x,y,y_b,y_h) of figure 3, we have the point $\mathbf{M_2}$ with coordinates **($\mathbf{x = 10}$, $\mathbf{y = 5}$, $\mathbf{y_b = 16}$, $\mathbf{y_h = 100}$)**, i.e., the point:

$$M_2 = (10,5,16,100) \qquad (3.20)$$

and so forth, for any value $x_i, y_i, (i = 1,2,3,...)$ that can be assigned to the variables x and y, we get a point M_i with coordinates:

$$M_i = \left(x_i, y_i, y_{b,i}, y_{h,i}\right) \qquad \mathbf{(3.21)}$$

Obviously, the points M_i of (3.21), all lie on the curve C of the given function of degree (3.15).

Conclusion:

A function of degree in two real variables, x and y, can be represented graphically on a 4-dimensional Euclidean space (orthogonal coordinate system (x,y,y_b,y_h)**) as a curve C, figure 3.**

C. n variable functions of degree

Suppose we have the function of degree:

$$Z_g = \frac{f_1(x_1,x_2,x_3,....x_n)}{f_2(x_1,x_2,x_3,....x_n)} = \frac{y_h}{y_b} \qquad (3.22)$$

with $x_1,x_2,x_3,....x_n$ variables.

In the manner above, we can consider functions of degree in **n** variables.

Conclusion:

A function of degree in n real variables, $x_1,x_2,x_3,....x_n$, can be represented graphically on a (n+2)-dimensional Euclidean space (orthogonal coordinate system ($\mathbf{x_1,x_2,x_3,...,y_b,y_h}$)) as a curve C, figure 4.

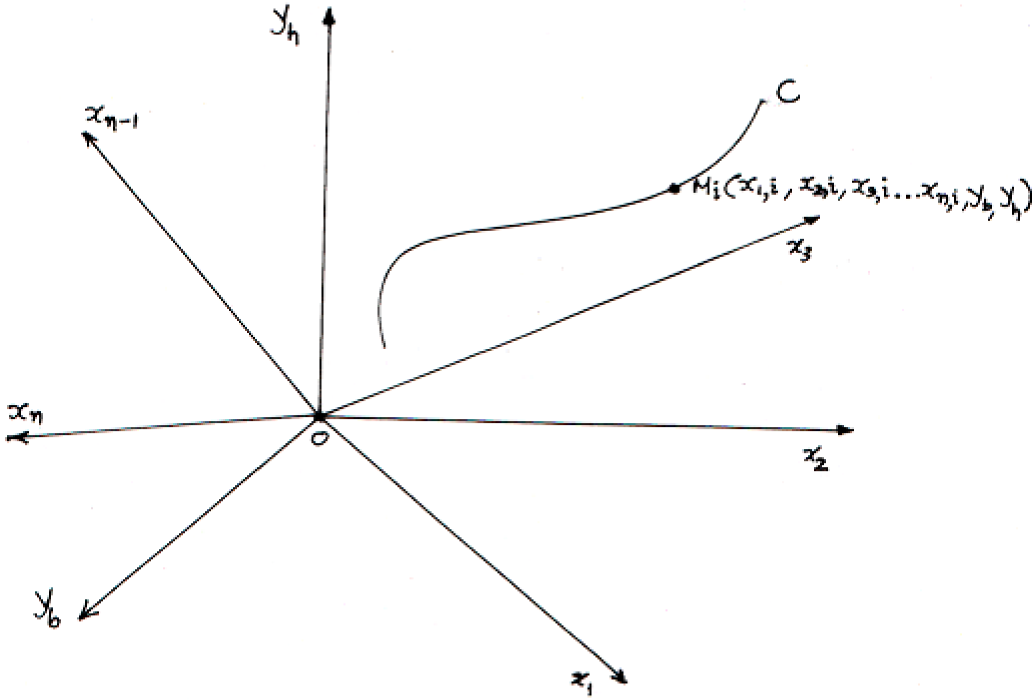


fig. 4

4. EQUATIONS OF DEGREE

For any function of degree:

$$y_g = f_g(x) \qquad (4.1)$$

in one variable, **x**, its conjugate equations are:

$$\text{and} \quad \left.\begin{aligned} y_h &= f_2(x) \\ y_b &= f_1(x) \end{aligned}\right\} \qquad (4.2)$$

Definition. We define the solutions of $f_g(x)$ in the category $\overset{1}{\mathrm{C}}$, to be the solution of the conjugate equations (4.2) for which:

$$\boxed{\begin{aligned} f_2(x) &= 1 \\ \text{and} \quad f_1(x) &= 0 \end{aligned}} \qquad (4.3)$$

The solutions of $f_2(x) = 1$ shall be called solutions of the hyperthetis and solutions of $f_1(x) = 0$ shall be called solutions of the base, of the given equation of degree.

Given the above, an equation of degree in one variable, x, whose solutions are required (in the category $\overset{1}{\mathbf{C}}$) will always be given in the form:

$$f_g(x)=\overset{1}{0}$$

Finally, equations (4.3) shall be called the conjugate equations of (4.1).

EXAMPLES

1. Solve the equation of degree:

$$\overset{2}{3}\cdot\overset{x}{x}-\overset{1}{3}=\overset{1}{0} \qquad (4.4)$$

SOLUTION

The given equation of degree can be rearranged to give:

$$\overset{2}{3}\cdot\overset{x}{x}-\overset{1}{3}=\overset{2\cdot x}{(3x)}-\overset{1}{3}=\overset{2\cdot x\cdot 1}{(3x-3)}=\overset{2x}{(3x-3)} \qquad (4.5)$$

Therefore, the conjugate equation of (4.4), from (4.5), are:

$$\left.\begin{array}{r} 2x=1 \\ \text{and } 3x-3=0 \end{array}\right\} \qquad (4.6)$$

The hyperthetis equation, **2x = 1**, is solved to give:

$$x_h=\frac{1}{2} \qquad (4.7)$$

which is the hyperthetis solution, and the base equation, **3x – 3 = 0**, is solved to give:

$$x_b=1 \qquad (4.8)$$

which is the base solution, of the given equation of degree (4.4).

Verification. Setting the hyperthetis solution (4.7) and the base solution (4.8) in the given equation of degree (4.4), should give us the neutral element $\overset{1}{\mathbf{0}}$.

Indeed,

$$\overset{2}{3}\cdot\overset{1/2}{1}-\overset{1}{3}=\overset{2\cdot\frac{1}{2}}{3}-\overset{1}{3}=\overset{2\cdot\frac{1}{2}\cdot 1}{(3-3)}=\overset{1}{0} \text{ QED.}$$

2. Solve the equation of degree:

$$\left(\overset{x}{x}\right)^{2} - \overset{5}{3}\left(\overset{x}{x}\right) + \overset{7}{2} = \overset{1}{0} \quad (4.5)$$

SOLUTION

The given equation of degree can be rearranged to give:

$$\left(\overset{x}{x}\right)^{2} - \overset{5}{3}\left(\overset{x}{x}\right) + \overset{7}{2} = \overset{x^2}{\left(x^2\right)} - \overset{5\cdot x}{\left(3\cdot x\right)} + \overset{7}{2} = \overset{x^2\cdot 5\cdot x\cdot 7}{\left(x^2-3x+2\right)} = \overset{35x^3}{\left(x^2-3x+2\right)}$$

whose conjugate equations are:

$$\left.\begin{array}{r} 35x^3 = 1 \\ \text{and } x^2-3x+2=0 \end{array}\right\} \quad (4.6)$$

Therefore, **the hyperthetis solutions of (4.5)** will be:

$$x_h = \sqrt[3]{1/35} \qquad (4.7)$$

and **the base solutions of (4.5)** will be:

$$\left.\begin{array}{r} x_{b_1} = 2 \\ \text{and } x_{b_2} = 1 \end{array}\right\} \quad (4.8)$$

Verification. Setting $x_h = \sqrt[3]{1/35}$ and $x_b = 2$ in the given equation of degree (4.5), we get:

$$\left(\overset{\sqrt[3]{1/35}}{2}\right)^{2} - \overset{5}{3}\left(\overset{\sqrt[3]{1/35}}{2}\right) + \overset{7}{2} = \overset{\sqrt[3]{(1/35)^2}}{4} - \overset{5}{3}\cdot \overset{\sqrt[3]{1/35}}{2} + \overset{7}{2} = \overset{35\sqrt[3]{(1/35)^3}}{\left(4-6+2\right)} = \overset{1}{0} \text{ QED.}$$

We proceed similarly with the other pair,

$$x_h = \sqrt[3]{1/35} \quad \text{and} \quad x_{b_2} = 1$$

which is again verified in the equation of degree (4.5)

3. Solve the equation of degree:

$$\left(x\right)^{\overset{5x}{2}} - \overset{5}{4} = \overset{1}{0} \qquad (4.9)$$

SOLUTION

The given equation of degree can be rearranged to give:

$$\overset{5x}{(x)}{}^{2} - \overset{5}{4} = \overset{(5x)^2}{\left(x^2\right)} - \overset{5}{4} = \overset{(5x)^2\cdot 5}{\left(x^2-4\right)} = \overset{125x^2}{(x^2-4)}$$

whose conjugate equations are:

$$\left.\begin{aligned} 125x^2 &= 1 \\ x^2 - 4 &= 0 \end{aligned}\right\} \qquad (4.10)$$

Therefore, the hyperthetis solutions of (4.9) are given by the first equation of (4.10) and are:

$$x_{h_1} = \sqrt{1/125} \quad \text{and} \quad x_{h_2} = -\sqrt{1/125} \qquad (4.11)$$

Similarly, the base solutions of (4.9) are given by the second equation of (4.10) and are:

$$x_{b_1} = 2 \quad \text{and} \quad x_{b_2} = -2 \qquad (4.12)$$

Therefore, the solutions of the given equation of degree (4.9) are (4.11) and (4.12). Obviously, verification of (4.9) can be achieved by taking one of the solutions (4.11) and one of the solutions (4.12), in any combination.

NOTEWORTHY EQUATIONS OF DEGREE

1. The equation of degree, of the form:

$$\overset{\kappa}{\alpha}\overset{x}{\left(x\right)} + \overset{\lambda}{\beta} = \overset{1}{0}$$

is a first order equation of degree in $\overset{x}{\mathbf{x}}$.

2. The equation of degree, of the form:

$$\overset{\kappa}{\alpha}\overset{x}{\left(x\right)}^{2} + \overset{\lambda}{\beta}\overset{x}{\left(x\right)} + \overset{\mu}{\gamma} = \overset{1}{0}$$

is a second order equation of degree in $\overset{x}{\mathbf{x}}$.

3. The equation of degree, of the form:

$$\overset{\kappa}{\alpha}\overset{x}{\left(x\right)}^{3} + \overset{\lambda}{\beta}\overset{x}{\left(x\right)}^{2} + \overset{\mu}{\gamma}\overset{x}{\left(x\right)} + \overset{\rho}{\delta} = \overset{1}{0}$$

is a third order equation of degree in $\overset{x}{\mathbf{x}}$.

4. The equation of degree, of the form:

$$\overset{\kappa}{\alpha_1}\left(\overset{x}{x}\right)^{\nu} + \overset{\lambda}{\alpha_2}\left(\overset{x}{x}\right)^{\nu-1} + \overset{\mu}{\alpha_3}\left(\overset{x}{x}\right)^{\nu-2} + \dots\dots + \overset{\rho}{\alpha_\nu} = \overset{1}{0}$$

is an ν^{th} order equation of degree in $\overset{x}{x}$.

The above equations of degree, without the second part, give us the equivalent polynomials of $\overset{x}{x}$.

NOTE: Besides the above mentioned equation of degree, many other interesting equations of degree exist, as the field of the mathematics of degree is unbounded and presents huge opportunities for mathematical research.

5. SIMULTANEOUS EQUATIONS OF DEGREE

EXAMPLE

1. Solve these simultaneous equation of degree in category $\overset{1}{C}$:

$$\left.\begin{array}{l} \overset{2x+1}{x} + \overset{3}{y} = \overset{-1}{x} - \overset{x}{y} \\ 3\overset{2}{x} + \overset{3x}{(y+1)} = \overset{y}{x} + \overset{x}{3} \end{array}\right\} \quad (5.1)$$

equations (5.1), can be solved as follows:

$$\left.\begin{array}{l} \overset{(2x+1)\cdot 3}{(x+y)} = \overset{-1\cdot x}{(x-y)} \\ \overset{1\cdot 2\cdot 3x}{(3x+y+1)} = \overset{y\cdot x}{(x+3)} \end{array}\right\} \quad (5.2) \text{ or}$$

$$\left.\begin{array}{l} \overset{6x+3}{(x+y)} = \overset{-x}{(x-y)} \\ \overset{6x}{(3x+y+1)} = \overset{xy}{(x+3)} \end{array}\right\} \quad (5.3)$$

From (5.3), we get the following sets of simultaneous equations:

1) Hyperthetis equations, i.e.,

$$\left.\begin{array}{l} 6x+3 = -x \\ 6x = xy \end{array}\right\} \quad (5.4)$$

and 2) Base equations, i.e.,

$$\left.\begin{aligned} x + y = x - y \\ 3x + y + 1 = x + 3 \end{aligned}\right\} \quad (5.5)$$

The solutions of the hyperthetis equations (5.4) are:

$$\left.\begin{aligned} x_h &= -\frac{3}{7} \\ y_h &= 6 \end{aligned}\right\} \quad (5.6)$$

Also, the solutions of the base equations (5.5) are:

$$\left.\begin{aligned} x_b &= 1 \\ y_b &= 0 \end{aligned}\right\} \quad (5.7)$$

The solutions (5.6) and (5.7) are the solutions of the given system (5.1).

Verification: Substituting (5.6) and (5.7) in (5.1), we get:

$$\left.\begin{aligned} 1^{2\cdot\left(-\frac{3}{7}\right)+1} + 0^{3} = 1^{-1} - 0^{-\frac{3}{7}} \\ 3\cdot 1^{2} + (0+1)^{3\cdot\left(-\frac{3}{7}\right)} = 1^{6} + 3^{-\frac{3}{7}} \end{aligned}\right\} \quad \text{or}$$

$$\left.\begin{aligned} 1^{\frac{1}{7}} + 0^{3} = 1^{-1} - 0^{-\frac{3}{7}} \\ 3\cdot 1^{2} + 1^{-\frac{9}{7}} = 1^{6} + 3^{-\frac{3}{7}} \end{aligned}\right\} \quad \text{or}$$

$$\left.\begin{aligned} (1+0)^{\frac{1}{7}\cdot 3} = (1-0)^{-1\cdot\left(-\frac{3}{7}\right)} \\ (3\cdot 1+1)^{1\cdot 2\cdot\left(-\frac{2}{7}\right)} = (1+3)^{6\cdot\left(-\frac{3}{7}\right)} \end{aligned}\right\} \quad \text{or}$$

$$\left.\begin{aligned} 1^{\frac{3}{7}} = 1^{\frac{3}{7}} \\ 4^{-\frac{18}{7}} = 4^{-\frac{18}{7}} \end{aligned}\right\} \quad \text{QED.}$$

6. DIFFERENTIATION OF FUNCTIONS OF DEGREE

Consider the function of degree:

$$y_g = f_g(x) \qquad (6.1)$$

in the variable, x, whose associated conjugate equations are:

$$\left.\begin{matrix} y_h = f_2(x) \\ y_b = f_1(x) \end{matrix}\right\} \qquad (6.2)$$

Definition. We define the first derivative of the function of degree $y_g = f_g(x)$ with respect to the variable x, denoted as $(y_g)'x$ as the first derivative with respect to x of the its associated conjugate equations, i.e.,

$$\boxed{(y_g)'x = \begin{matrix} (y_h)'x \\ (y_b)',x \end{matrix}} \qquad (6.4)$$

Based on the above definition, we can have higher order derivatives with respect to x in a similar manner. The same obviously holds for functions of degree with more than one variable.

EXAMPLES

1. Find the first derivative with respect to x of the following equation of degree:

$$y_g = \overset{2}{3}\cdot\overset{x}{x} - \overset{x}{5} + \overset{6}{x} \qquad (6.5)$$

SOLUTION

The given equation of degree in its final state:

$$y_g = \overset{2}{3}\cdot\overset{x}{x} - \overset{x}{5} + \overset{6}{x} = \overset{2\cdot x\cdot x\cdot 6}{(3x - 5 + x)} = \overset{12x^2}{(4x - 5)}$$

the associated **conjugate equations** are:

$$\left.\begin{matrix} y_h = 12x^2 \\ y_b = 4x - 5 \end{matrix}\right\} \qquad (6.6)$$

whose first derivatives w.r.t. x are (respectively):

$$\left.\begin{matrix} (y_h)'x = 24x \\ (y_b)'x = 4 \end{matrix}\right\} \qquad (6.7)$$

These equations (6.7) are the required first derivatives w.r.t. x for the given equation of degree (6.5).

2. Find the second derivative w.r.t. x of the following equation of degree:

$$y_b = \overset{2}{x^3} + \overset{x^4}{5} + \overset{4}{2} \qquad (6.8)$$

SOLUTION

The given equation of degree in its final state:

$$y_g = \overset{2}{x^3} + \overset{x^4}{5} + \overset{4}{2} = \overset{2^3}{\left(x^3\right)} + \overset{x^4}{5} + \overset{4}{2} = \overset{8}{\left(x^3\right)} + \overset{x^4}{5} + \overset{4}{2}$$

$$\text{or } y_g = \overset{32x^4}{\left(x^3 + 7\right)}$$

the associated **conjugate equations** are:

$$\left.\begin{aligned} y_h &= 32x^4 \\ y_b &= x^3 + 7 \end{aligned}\right\} \qquad (6.9)$$

whose first derivatives w.r.t. x are (respectively):

$$\left.\begin{aligned} (y_h)'x &= 128x^3 \\ (y_b)'x &= 3x^2 \end{aligned}\right\} \qquad (6.10)$$

the second derivatives w.r.t. x are (respectively):

$$\left.\begin{aligned} (y_h)''x &= 384x^2 \\ (y_b)''x &= 6x \end{aligned}\right\} \qquad (6.11)$$

These equations (6.11) are the required second derivatives w.r.t. x (i.e., $\left(y_g\right)'' x$) for the given equation of degree (6.8).

FUNCTIONS OF DEGREE

Consider the function of degree:

$$y_g = f_g(x_1, x_2, x_3, \ldots. x_v) \qquad (6.12)$$

in the variables, $x_1, x_2, x_3, \ldots. x_v$, whose associated **conjugate equations** are:

$$\left.\begin{aligned} y_h &= f_2(x_1, x_2, x_3, \ldots x_\nu) \\ y_b &= f_1(y_h = f_2(x_1, x_2, x_3, \ldots x_\nu)) \end{aligned}\right\} \qquad (6.13)$$

Based on what was mentioned above, the first derivative of the given function with respect to one of the variables, x_κ, i.e.,

$$(y_g)'x_\kappa$$

is the first derivative w.r.t. x_κ of the equations of degree (6.13), i.e.:

$$(y_g)'_{x_\kappa} = \frac{(y_h)'_{x_\kappa}}{(y_b)'_{x_\kappa}} \qquad (6.14)$$

Similarly, the second derivative will be:

$$(y_g)''_{x_\kappa} = \frac{(y_h)''_{x_\kappa}}{(y_b)''_{x_\kappa}} \qquad (6.15)$$

and in general, the ν^{th} derivative will be:

$$(y_g)^{(\nu)}_{x_\kappa} = \frac{(y_h)^{(\nu)}_{x_\kappa}}{(y_b)^{(\nu)}_{x_\kappa}} \qquad (6.16)$$

In the manner mentioned above, we can find the partial derivatives of functions of degree with more than one variable.

7. INTEGRATION OF FUNCTIONS OF DEGREE

Consider the function of degree:

$$y_g = f_g(x) \qquad (7.1)$$

with variable x.

Definition. We define as integration of degree that of thee equation of degree (7.1) and we denote it by:

$$\int f_g(x) \cdot dx$$

the integrals of its associated conjugate equations are:

$$\boxed{\int y_g(x)\cdot dx = \frac{\int f_h(x)\cdot dx}{\int f_b(x)\cdot dx}} \quad (7.2)$$

Where $f_h(x)$ and $f_b(x)$, are the associated conjugate equations of $f_g(x)$.
Also, the integrals (7.2) will be called the conjugate integrals of the equation of degree (7.1).

In the same manner, we can have double, triple, and so on, integrals of different equations of degree with more than one variable.

EXAMPLES

1. Find the integral of degree:

$$\int [(\overset{x^2-1}{x} + \overset{3}{(x^2)} + \overset{2}{5}\cdot x + \overset{7}{4})]\cdot dx \quad (7.3)$$

The integral of degree (7.3), in its final state:

$$\int [(\overset{x^2-1}{x} + \overset{3}{(x^2)} + \overset{2}{5}\cdot x + \overset{7}{4})]\cdot dx = \int \left(x + x^2 + 5x + 4\right)^{(x^2-1)\cdot 3\cdot 2\cdot 7}\cdot dx = \int \left(x^2 + 6x + 4\right)^{42x^2-42} dx$$

So the conjugate integrals of the integral (7.3) are by definition:

$$\left.\begin{array}{l} \int (42x^2 - 42)\cdot dx \\ \int (x^2+6x+4)\cdot dx \end{array}\right\} \quad (7.4)$$

In (7.4), the first integral is the **hyperthetis integral** and the second the **base integral**.

Working out (7.4) we get:

$$\int (42x^2 - 42)\cdot dx = 14x^3 - 42x + c_1$$

and:

$$\int (x^2+6x+4)\cdot dx = \frac{x^3}{3} + 3x^2 + 4x + c_2$$

where $\mathbf{c_1}$ and $\mathbf{c_2}$ are the integration constants.

Therefore, solution of the given integral (7.3) comes from the following equations:

$$\left.\begin{array}{l} 14x^3 - 42x + c_1 \\ \dfrac{x^3}{3} - 3x^2 + 4x + c_2 \end{array}\right\} \quad (7.5)$$

2. Find the integral of degree:

$$\int (\overset{3x}{x^2} + \overset{x}{x^5} + \overset{x}{1}) \cdot dx \qquad (7.6)$$

The integral of degree (7.6), in its final state:

$$\int (\overset{3x}{x^2} + \overset{x}{x^5} + \overset{x}{1}) \cdot dx = \int \overset{(3x)^2}{[x^2]} + (\overset{x^5}{x^5}) + \overset{x}{1}]\, dx = \int \overset{9x^2 \cdot x^5 \cdot x}{\left(x^2 + x^5 + 1\right)} dx = \int \overset{9x^8}{\left(x^5 + x^{2+1}\right)} \cdot dx \quad (7.7.)$$

So the conjugate integrals of the integral (7.7) are:

$$\left.\begin{array}{l} \int 9x^8 \cdot dx \\ \int \left(x^5 + x^2 + 1\right) \cdot dx \end{array}\right\} \qquad (7.8)$$

From the first integral (7.8), we have:

$$\int 9x^8 \cdot dx = x^9 + c_1$$

and from the second integral (7.8), we have:

$$\int \left(x^5 + x^2 + 1\right) \cdot dx = \frac{x^6}{6} + \frac{x^3}{3} + x + c_2$$

where $\mathbf{c_1}$ and $\mathbf{c_2}$ are the integration constants.

Therefore, solution of the given integral (7.6) comes from the following equations (7.9):

$$\left.\begin{array}{c} x^9 + c_1 \\ \dfrac{x^6}{6} + \dfrac{x^3}{3} + x + c_2 \end{array}\right\} (7.9)$$

8. DIFFERENTIAL EQUATIONS OF DEGREE

Definition. An v^{th} order differential equation, in category $\overset{1}{C}$, is an equation of degree an unknown function, y_g, with its variable, x, and its derivatives up to order v, namely: $y_g : y'_g, y''_g, \ldots. y_g^{(v)}$. I.e., an equation of the form:

$$f(x, y_g, y'_g, y''_g, \ldots. y_g^{(v)}) = \overset{1}{0} \qquad \textbf{(8.1)}$$

The solution, or integral of (8.1) is two equations:

$$\boxed{\begin{matrix} y_h(x) \\ y_b(x) \end{matrix}} \quad \textbf{(8.2)}$$

The first is the hyperthetis solution and the second the base solution, and they satisfy (8.1) by definition.

EXAMPLES

1. Solve the differential equation of degree:

$$y''_g \overset{x}{+} 3\cdot y'_g \overset{2}{+} 5\cdot y_g \overset{1}{+} x \overset{1}{=} 0 \qquad (8.3)$$

SOLUTION

The given differential equation of degree is obviously second order.
For its solution we proceed as follows:

$$\left.\begin{aligned} y_g &= \begin{matrix} y_h \\ y_b \end{matrix} \\ y'_g &= \begin{matrix} y'_h \\ y'_b \end{matrix} \\ y''_g &= \begin{matrix} y''_h \\ y''_b \end{matrix} \end{aligned}\right\} \qquad (8.4)$$

Substituting (8.4) in (8.3) we get:

$$\overset{y''_h}{y''_b} \overset{x}{+} 3\cdot \overset{y'_h}{y'_b} \overset{2}{+} 5\cdot \overset{y_h}{y_b} \overset{1}{+} x \overset{1}{=} 0 \Rightarrow \overset{y''_h\cdot x\cdot y'_h\cdot 2\cdot y_h\cdot 1}{\left(y''_b + 3y'_b + 5y_b + x\right)} = \overset{y''_h\cdot y'_h\cdot y_h\cdot 2x}{\left(y''_b + 3y'_b + 5y_b + x\right)} = \overset{1}{0}$$

The above equation is associated with the following conjugate differential equations of (8.3):

$$\left.\begin{aligned} y''_h\cdot y'_h\cdot y_h\cdot 2x = 1 \\ y''_b + 3y'_b + 5y_b + x = 0 \end{aligned}\right\} \qquad (8.5)$$

The first is the differential equation of the hyperthetis and the second the differential equation of the base of the differential equation of degree (8.3).

Therefore, by solving the differential equations (8.5) in the familiar manner, we get the required solution of the differential equation of degree (8.3).

Verification. Solving $y_h(x)$ and $y_b(x)$ with reference to (8.4), we continue as usual to find, y_g, y'_g, y''_g. Substituting these values in the given (8.4), we should be able to rearrange to get $\overset{1}{0}$.

9. SEQUENCES OF DEGREE

The function:

$$a_{g_\nu} = f_g(\nu) \qquad (9.1)$$

where $a_{g_\nu} = \genfrac{}{}{0pt}{}{a_{h_\nu}}{a_{b_\nu}}$ **and v = 1, 2, 3, …**

constitutes a sequence of degree with general term, a_{g_ν} **.**

a_{h_ν} **shall be called the hyperthetis general term and** a_{b_ν} **the base general term of the sequence of degree (9.1).**

In addition, a sequence of degree is by definition equivalent to the hyperthetis sequence and the base sequence, which shall be called, **conjugate sequences of the sequence of degree (9.1), i.e.:**

$$\boxed{a_{g_\nu} = f_g(\nu) = \begin{matrix} a_{h_\nu} = f_h(\nu) \\ a_{b_\nu} = f_b(\nu) \end{matrix}} \qquad (9.2)$$

The study therefore of a sequence of degree is directly linked to the study of its conjugate sequences.
In general then, we can say that a sequence of degree, e.g. (9.1), converges in the hyperthetis or converges in the base or converges in both or diverges in the hyperthetis and so on, depending on the conjugate sequences (9.10).

EXAMPLES

1. Find the first three terms of the following sequence:

$$a_{g_\nu} = \left(\nu^3\right)^{\nu^2-1} + 4^{\nu^3} + 5^{7} \qquad (9.3)$$

SOLUTION

The given sequence of degree (9.3) can be rearranged to become:

$$a_{g_v} = \overset{v^2-1}{\left(v^3\right)} + \overset{v^3}{4} + \overset{7}{5} = \overset{(v^2-1)\cdot v^3\cdot 7}{\left(v^3+4+5\right)} = \overset{7v^5-7v^3}{(v^3+9)}$$

Consequently, the conjugate sequences of (9.3), are:

$$\left.\begin{aligned} a_{h_v} &= 7v^5 - 7v^3 \\ a_{b_v} &= v^3 + 9 \end{aligned}\right\} \qquad (9.4)$$

So, for **v = 1 the first of (9.4)** gives us:

$$a_{h_1} = 7 \cdot 1^5 - 7 \cdot 1^3 = 7 - 7 = 0$$

for **v = 2**:

$$a_{h_2} = 7 \cdot 2^5 - 7 \cdot 2^3 = 224 - 56 = 168$$

and for **v = 3**:

$$a_{h_3} = 7 \cdot 3^5 - 7 \cdot 3^3 = 1701 - 63 = 1638$$

Also, **the second of (9.4),** for **v = 1** gives us:

$$a_{b_1} = 1^3 + 9 = 1 + 9 = 10$$

for **v = 2**:

$$a_{b_2} = 2^3 + 9 = 8 + 9 = 17$$

and for **v = 3**:

$$a_{b_3} = 3^3 + 9 = 27 + 9 = 36$$

Therefore the first three terms of the given sequence of degree (9.3) are:

$$\left.\begin{aligned} &a_{h_1} = 0, a_{h_2} = 168, a_{h_3} = 1638 \\ &a_{b_1} = 10, a_{b_2} = 17, a_{b_3} = 36 \end{aligned}\right\} \qquad (9.5)$$

2. Find the 17th term of the following sequence:

$$a_{g_\nu} = \overset{-\nu}{\nu^2} + \overset{\nu}{\nu^\nu} \quad (9.6)$$

SOLUTION

The given sequence of degree (9.6) can be rearranged to become:

$$a_{g_\nu} = \overset{-\nu}{\nu^2} + \overset{\nu}{\nu^\nu} = \overset{(-\nu)^2}{\left(\nu^2\right)} + \overset{(\nu)^\nu}{\left(\nu^\nu\right)} = \overset{\nu^2 \cdot \nu^\nu}{\left(\nu^2 + \nu^\nu\right)}$$

$$\text{or} \quad a_{g_\nu} = \overset{\nu^{(\nu+2)}}{\left(\nu^\nu + \nu^2\right)}$$

Consequently, the conjugate sequences of (9.6), are:

$$\left.\begin{aligned} a_{h_\nu} &= \nu^{(\nu+2)} \\ a_{b_\nu} &= \nu^\nu + \nu^2 \end{aligned}\right\} \quad (9.7)$$

So for **v = 17**, we get:

$$a_{h_{17}} = 17^{(17+2)} = 17^{19}$$

and:

$$a_{b_{17}} = 17^{17} + 17^2 = 17^2(17^{15} + 1)$$

$$\text{or} \quad a_{b_{17}} = 289(17^{15} + 1)$$

Therefore, the 17th term of the given sequence of degree (9.6), is

$$\left.\begin{aligned} a_{h_{17}} &= 17^{19} \\ a_{b_{17}} &= 289(17^{15} + 1) \end{aligned}\right\} \quad (9.8)$$

10. SERIES OF DEGREE

Following the same reasoning used for sequences of degree, we can consider series of degree.

Consider the following series of degree:

$$\sum_{n=1}^{\infty} {}_g \overset{n^3+3}{\frac{1}{\overset{n}{\left(n^2\right)} \overset{5}{+1}}} \quad (10.1)$$

The series (10.1) can be rearranged to become:

$$\sum_{n=1}^{\infty} {}_g \overset{n^3+3}{\frac{1}{\overset{n}{\left(n^2\right)} \overset{5}{+1}}} = \sum_{n=1}^{\infty} {}_g \overset{n^3+3}{\frac{1}{\overset{n\cdot 5}{\left(n^2+1\right)}}} = \sum_{n=1}^{\infty} {}_g \overset{n^3+3}{\frac{1}{\overset{5n}{\left(n^2+1\right)}}} = \sum_{n=1}^{\infty} {}_g \overset{(n^3+3):5n}{[1:(n^2+1)]}$$

So we have:

$$\left.\begin{array}{l}\sum_{n=1}^{\infty}{}_{h}\dfrac{n^3+3}{5n}\\ \sum_{n=1}^{\infty}{}_{b}\dfrac{1}{n^2+1}\end{array}\right\} \qquad (10.2)$$

The relationships (10.2) are, as usual, the conjugate series of the given series of degree (10.1). The first of equations (10.2) is the hyperthetis series and the second, the base series of the given series of degree (10.1).

Finally, study of the given series of degree (10.1) depends on the study of the conjugate series (10.2), regarding its convergence, divergence etc.

11. THE FOUR BASIC OPERATIONS ON THE CONJUGATE FUNCTIONS

Consider the function of degree:

$$y_g = f_g(x) \qquad (11.1)$$

in the variable x, whose conjugate equations are:

$$\left.\begin{array}{l} y_h = f_h(x) \\ y_b = f_b(x)\end{array}\right\} \qquad (11.2)$$

Definition. **a. The function** $y_{g,+}$, i.e.:

$$y_{g,+} = f_b(x) + f_h(x)$$

will be called conjugate addition of (11.1).

b. The function $y_{g,-}$, i.e.:

$$y_{g,\text{-}} = f_b(x) - f_h(x)$$

will be called conjugate subtraction of (11.1).

c. The function $y_{g,\cdot}$, i.e.:

$$y_{g,\cdot} = f_b(x) \cdot f_h(x)$$

will be called conjugate multiplication of (11.1).

d. The function $y,_{g:}$, i.e.:

$$y_{g,:} = f_b(x) : f_h(x)$$

will be called conjugate division of (11.1).

NOTEWORTHY OBSERVATION

Suppose that (11.1) is for example an equation of degree, a differential equation of degree, an integral equation of degree, a sequence of degree and so forth, then we would have the conjugate operations (a), (b), (c), (d).

The conjugate forms of degree, in the basic operations just witnessed, play a vital role in the mathematics of degree as they further deepen the field open to research. This becomes immediately obvious with the following example:

Lets take an equation of degree for example. After working out its final form, the equation is «split» into its two conjugate equations (the hyperthetis equation and the base equation) whose solutions give us the solution of the given equation of degree.

With the introduction of the basic operations on the conjugate functions - **(a), (b), (c), (d)** - new equations result, with new solutions that enable research into their possible relation.

The same holds for sequences of degree, series of degree etc...

EXAMPLE 1

Given the function of degree:

$$Z_g = \overset{x+y}{3xy} + \overset{x}{2} \qquad (11.3)$$

in the variables **x** and **y**.

Find $Z_{g,+}$, $Z_{g,-}$, $Z_{g,\cdot}$, $Z_{g,:}$

SOLUTION

The function of degree (11.3) can be rearranged to give:

$$Z_g = \overset{(x+y)x}{(3xy+2)} = \overset{x^2+xy}{(3xy+2)} \qquad (11.4)$$

whose conjugate equations are:

$$\left.\begin{array}{l} Z_h = x^2 + xy \\ Z_b = 3xy + 2 \end{array}\right\} \qquad (11.5)$$

Consequently:

1. $Z_{g,+} = Z_b + Z_h = (3xy+2) + (x^2 + xy) = 4xy + 2 + x^2$

2. $Z_{g,-} = Z_b - Z_h = (3xy+2) - (x^2+xy) = 2xy + 2 - x^2$

3. $Z_{g,\cdot} = Z_b \cdot Z_h = (3xy+2)\cdot(x^2+xy)$

4. $Z_{g,:} = \dfrac{Z_b}{Z_h} = \dfrac{3xy+2}{x^2+xy}$

EXAMPLE 2

Given the sequence of degree:

$$a_{g_\nu} = \left(\sqrt[\nu]{\nu} \quad -\nu\right)^{\nu}$$

Find the limit as **(v → ∞)**, of the sequence that results from its conjugate addition.

SOLUTION

From the given sequence of degree, we have:

$$a_{h_\nu} = \nu \qquad (\textit{hyperthetis sequence})$$

$$a_{b_\nu} = \sqrt[\nu]{\nu} - \nu \quad (\textit{base sequence})$$

So, the sequence $a_{g_{\nu,+}}$ that results from the conjugate addition of the given sequence is:

$$a_{g_{\nu,+}} = a_{b_\nu} + a_{h_\nu} = \sqrt[\nu]{\nu} - \nu + \nu = \sqrt[\nu]{\nu}\text{, i.e.}$$

$$a_{g_{\nu,+}} = \sqrt[\nu]{\nu}$$

Of course, the limit of $a_{g_{\nu,+}} = \sqrt[\nu]{\nu}$ as **(v → ∞)** is:

$$\lim_{\nu\to\infty} \sqrt[\nu]{\nu} = 1$$

which is the required solution.

12. PRIME NUMBERS OF DEGREE

In the mathematics known hitherto, the numbers,

1, 5, 7, 13, 17, etc...

are prime numbers.

But in the mathematics of degree, because the numbers we use are numbers of different degree, we must give the following definitions:

Definition. A number of degree $\overset{\kappa}{a}$ shall be called, base prime, when its base, α, is a prime number.

Definition. A number of degree $\overset{\kappa}{a}$ shall be called, hyperthetis prime, when its hyperthetis, κ, is a prime number.

Definition. A number of degree $\overset{\kappa}{a}$ shall be called, complete prime, when its hyperthetis, κ, and its base, α, are prime numbers.

So, based on the first definition regarding the base, base prime numbers are:

$$\overset{7}{3},\ \overset{2}{7},\ \overset{5}{17},\ \overset{4}{13},\ \text{etc...}$$

Based on the second definition regarding the hyperthetis, hyperthetis prime numbers are:

$$\overset{3}{4},\ \overset{7}{3},\ \overset{5}{13},\ \overset{17}{8},\ \text{etc...}$$

Finally, based on the third definition, complete prime numbers are:

$$\overset{7}{17},\ \overset{5}{13},\ \overset{19}{7},\ \overset{7}{11},\ \text{etc...}$$

Note: The numbers of degree $\overset{\kappa}{a}$ and $\overset{a}{\kappa}$, are called inverse numbers of degree. Now, if $\overset{\kappa}{a}$ is a complete prime, then its inverse $\overset{a}{\kappa}$ will also be a complete prime.

Finally, as we observe, in the mathematics of degree, the mathematical field of «Number theory of degree» is broader than the equivalent field of «Number theory» of classical mathematics, hitherto known.

13. EQUAL AND UNEQUAL NUMBERS OF DEGREE

Definition. Two numbers od degree $\overset{\kappa}{\alpha}$ and $\overset{\lambda}{\beta}$, are equal, i.e.:

$$\overset{\kappa}{\alpha} = \overset{\lambda}{\beta}$$

when:

$$\left.\begin{aligned} \kappa &= \lambda \\ and\ \alpha &= \beta \end{aligned}\right\} \qquad (13.1)$$

Definition. A number of degree $\overset{\kappa}{\alpha}$ is bigger than another number of degree $\overset{\lambda}{\beta}$, i.e.:

$$\overset{\kappa}{\alpha} > \overset{\lambda}{\beta}$$

when:

$$\left.\begin{aligned} \kappa &= \lambda \\ and\ \alpha &> \beta \end{aligned}\right\} \qquad (13.2)$$

Definition. A number of degree $\overset{\kappa}{\alpha}$ is smaller than another number of degree $\overset{\lambda}{\beta}$, i.e.:

$$\overset{\kappa}{\alpha} < \overset{\lambda}{\beta}$$

when:

$$\left.\begin{aligned} \kappa &= \lambda \\ and\ \alpha &< \beta \end{aligned}\right\} \qquad (13.3)$$

EXAMPLE

1. Solve, in category $\overset{1}{C}$, the inequality:

$$\overset{\kappa^2}{(x^2+1)} + \overset{1}{2} > \overset{\kappa}{x^2+x} \qquad (13.4)$$

SOLUTION

From inequality (13.4), we get:

$$\overset{\kappa^2\cdot 1}{\left(x^2+1+2\right)} > \overset{1\cdot\kappa}{\left(x^2+x\right)} \text{ or}$$

$$\overset{\kappa^2}{\left(x^2+3\right)} > \overset{\kappa}{\left(x^2+x\right)} \qquad (13.5)$$

From (13.5), we must have:

$$\left.\begin{aligned} \kappa^2 &= \kappa \\ and\ x^2+3 &> x^2+x \end{aligned}\right\} \Rightarrow \ (13.6)$$

$$\left.\begin{aligned} \kappa &= 1 \\ and\ x &< 3 \end{aligned}\right\} \qquad (13.7)$$

which is the required solution.

Verification: From (13.7), take for example **κ = 1** and **x = 2**, then the inequality (13.4) gives:

$$\overset{1^2}{\left(2^2+1\right)} + \overset{1}{2} > 2^2 + \overset{1}{2} \quad \text{or}$$

$$\overset{1^2\cdot 1}{(2^2+1+2)} > \overset{1\cdot 1}{\left(2^2+2\right)} \quad \text{or}$$

$$\overset{1}{7} > \overset{1}{6} \quad \text{or}$$

$$7 > 6 \quad \text{QED.}$$

DIFFERENT TYPES OF NUMBER OF DEGREE

If $\overset{\kappa}{a}$ is a number of degree, then:

Definition: The number:

$$n_{\alpha+\kappa} = \alpha + \kappa$$

is called, an **additive** of the number of degree $\overset{\kappa}{a}$.

Definition: The number:

$$n_{\alpha-\kappa} = \alpha - \kappa$$

is called, a **subtractive** of the number of degree $\overset{\kappa}{a}$.

Definition: The number:

$$n_{\alpha\cdot\kappa} = \alpha \cdot \kappa$$

is called, a **multiplicative** of the number of degree $\overset{\kappa}{a}$.

Definition: The number:

$$n_{\alpha:\kappa} = \alpha : \kappa$$

is called, a **divisional** of the number of degree $\overset{\kappa}{a}$.

Definition: Two numbers $\overset{\kappa}{a}$ and $\overset{\lambda}{\beta}$, are incompatible $\overset{\kappa}{a}\#\overset{\lambda}{\beta}$, when:

$$\kappa \neq \lambda$$

where, **α** and **β** can be equal or not.

In addition, if **A** and **B** are the moduli of the numbers of degree $\overset{\kappa}{a}$ and $\overset{\lambda}{\beta}$**,** i.e.:

$$\mathrm{A} = \sqrt{\alpha^2 + \kappa^2}$$

$$\mathrm{B} = \sqrt{\beta^2 + \lambda^2}$$

and:

- **a. A > B**
- **b. A = B**
- **c. A < B**

then, from the two incompatible numbers $\overset{\kappa}{a}$ and $\overset{\lambda}{\beta}$**,** for the above three cases **(a), (b), (c),** $\overset{\kappa}{a}$ shall be called,

- **a.** «In modulus», bigger than $\overset{\lambda}{\beta}$
- **b.** «In modulus», equal to $\overset{\lambda}{\beta}$
- **c.** «In modulus», smaller than $\overset{\lambda}{\beta}$

14. COMPLEX NUMBERS OF DEGREE BASIC CONCEPTS AND DEFINITIONS

The notation,

$$\overset{z_k}{z} \qquad \textbf{(14.1)}$$

where z and $\overset{z_k}{z}$ **, are complex numbers of the form:**

$$z = x + iy$$
$$\text{and } z_\kappa = x_\kappa + iy_\kappa$$

(x, y, x_κ**,** y_κ **real numbers)**

constitutes a complex number of degree, with base, z, and hyperthetis, z_κ**.**

The hyperthetis, z_κ**, represents the complex degree to which the complex number of degree (14.1) belongs.**

So, the hitherto known complex numbers of degree, e.g.,

$$\mathbf{4+2i},\ \ \mathbf{5-2i},\ \ \mathbf{-3+6i}\ \text{etc...} \qquad (14.2)$$

are by definition complex numbers of degree 1, category $\overset{1}{\mathbf{C}}$. That is, the hyperthetis of (14.2) is considered to have a value of unity, which is omitted for convenience.

Its worth noting that the hyperthetis, z_κ, can be a complex number, without its real or imaginary part.

AXIOMATIC FOUNDATION OF COMPLEX NUMBERS OF DEGREE, CATEGORY $\overset{1}{\mathbf{C}}$.

15. BASIC OPERATIONS ON COMPLEX NUMBERS OF DEGREE

1. Addition

$$\overset{z_\kappa}{z_1}+\overset{z_\lambda}{z_2}=\overset{z_\kappa \cdot z_\lambda}{\left(z_1+z_2\right)}$$

2. Subtraction

$$\overset{z_\kappa}{z_1}-\overset{z_\lambda}{z_2}=\overset{z_\kappa \cdot z_\lambda}{\left(z_1-z_2\right)}$$

3. Multiplication

$$\overset{z_\kappa}{z_1}\cdot\overset{z_\lambda}{z_2}=\overset{z_\kappa \cdot z_\lambda}{\left(z_1\cdot z_2\right)}$$

4. Division

$$\overset{z_\kappa}{z_1}:\overset{z_\lambda}{z_2}=\overset{z_\kappa : z_\lambda}{\left(z_1 : z_2\right)}$$

If $\overset{z_\kappa}{z_1}$, $\overset{z_\lambda}{z_2}$, $\overset{z_\mu}{z_3}$ are elements of the set $\mathbf{C_g}$ of complex numbers of degree, category $\overset{1}{\mathbf{C}}$, then:

a. $\overset{z_\kappa}{z_1}+\overset{z_\lambda}{z_2}$ and $\overset{z_\kappa}{z_1}\cdot\overset{z_\lambda}{z_2}$ $\overset{\kappa}{a_1}\cdot\overset{\lambda}{a_2}$ belong to $\mathbf{C_g}$ (closure property).

b. $\overset{z_\kappa}{z_1}+\overset{z_\lambda}{z_2}=\overset{z_\lambda}{z_2}+\overset{z_\kappa}{z_1}$ (commutative property).

c. $\overset{z_\kappa}{z_1}+(\overset{z_\lambda}{z_2}+\overset{z_\mu}{z_3})=(\overset{z_\kappa}{z_1}+\overset{z_\lambda}{z_2})+\overset{z_\mu}{z_3}$ (associative property).

d. $\overset{z_\kappa}{z_1}\cdot(\overset{z_\lambda}{z_2}\cdot\overset{z_\mu}{z_3}) = (\overset{z_\kappa}{z_1}\cdot\overset{z_\lambda}{z_2})\cdot\overset{z_\mu}{z_3}$ (associative property).

e. $\overset{z_\kappa}{z_1}\cdot\overset{z_\lambda}{z_2} = \overset{z_\lambda}{z_2}\cdot\overset{z_\kappa}{z_1}$ (commutative property).

f. $\overset{z_\kappa}{z_1}\cdot(\overset{z_\lambda}{z_2}+\overset{z_\mu}{z_3}) = \overset{z_\kappa}{z_1}\cdot\overset{z_\lambda\cdot z_\mu}{(z_2+z_3)}$ and not $\overset{z_\kappa}{z_1}\cdot(\overset{z_\lambda}{z_2}+\overset{z_\mu}{z_3}) = \overset{z_\kappa}{z_1}\cdot\overset{z_\lambda}{z_2}+\overset{z_\kappa}{z_1}\cdot\overset{z_\mu}{z_3}$ (non-distributive property).

g. $\overset{z_\kappa}{z_1}+\overset{1}{0} = \overset{1}{0}+\overset{z_\kappa}{z_1} = \overset{z_\kappa}{z_1}$

$\overset{1}{0}$ is called the identity element of addition.

h. $\overset{z_\kappa}{z_1}\cdot\overset{1}{1} = \overset{1}{1}\cdot\overset{z_\kappa}{z_1} = \overset{z_\kappa}{z_1}$

$\overset{1}{1}$ is called the identity element of multiplication.

i. For every complex number of degree $\overset{z_\kappa}{z_1}$ there is only one complex number of degree $\overset{z_\rho}{z}$ of $\mathbf{C_g}$, such that:

$$\overset{z_\kappa}{z_1}+\overset{z_\rho}{z} = \overset{\mathbf{1}}{\mathbf{0}}$$

$\overset{z_\rho}{z}$ is called the additive inverse of $\overset{z_\kappa}{z_1}$ $\overset{\kappa}{a}_1$ and is denoted by:

$$\overset{1/z_\kappa}{-z_1}\text{, where } z_\rho = \frac{1}{z_\kappa}.$$

j. For every complex number of degree $\overset{z_\kappa}{z_1} \neq \overset{1}{0}$ there is only one complex number of degree $\overset{z_\rho}{z}$ of $\mathbf{C_g}$, such that:

$$\overset{z_\kappa}{z_1}\cdot\overset{z_\rho}{z} = \overset{z_\rho}{z}\cdot\overset{z_\kappa}{z_1} = \overset{1}{1}$$

$\overset{z_\rho}{z}$ is called the multiplicative inverse of $\overset{z_\kappa}{z_1}$ $\overset{\kappa}{a}_1$ and is denoted by:

$$\overset{1/z_\kappa}{\left(\frac{1}{\overset{z_\rho}{z}}\right)}$$

k. Every complex number of degree, category $\overset{1}{\mathbf{C}}$, when a term is moved from one side of the equation to the other, then, the sign of the base changes and the hyperthetis takes the value of its inverse.

IDENTITIES AND DEFINITIONS

The identities and definitions developed for real numbers of degree are also valid for complex numbers of degree, so to avoid unnecessary repetition, they will be omitted here.

EXAMPLES

1. Calculate the sum:

$$A = \overset{2-i}{\left(3+2i\right)} + \overset{2+i}{\left(5-2i\right)} + \overset{3i}{(10+3i)}$$

As we know, we get:

$$A = \overset{2-i}{\left(3+2i\right)} + \overset{2+i}{\left(5-2i\right)} + \overset{3i}{\left(10+3i\right)} = \overset{(2-i)(2+i)\cdot 3i}{\left(3+2i+5-2i+10+3i\right)} = \overset{15i}{\left(18+3i\right)}$$

2. Calculate the product:

$$A = \overset{3}{\left(3+2i\right)}\overset{i}{\left(3-2i\right)}$$

We get:

$$A = \overset{3}{\left(3+2i\right)}\overset{i}{\left(3-2i\right)} = \overset{3\cdot i}{\left(3^2+2^2\right)} = \overset{3i}{13}$$

3. Calculate the ratio:

$$A = \frac{\overset{i}{(3i)}}{\overset{i}{3}}$$

We get:

$$A = \frac{\overset{i}{(3i)}}{\overset{i}{3}} = \overset{i/i}{\left(\frac{3i}{3}\right)} = \overset{1}{i} = i$$

4. Prove that:

$$\overset{i^2}{e}\cdot\overset{i^2}{e^{-1}} = 1$$

We get:

$$\overset{i^2}{e}\cdot\overset{i^2}{e^{-1}} = \overset{i^4}{e^{0}} = \overset{1}{1} = 1$$

5. Calculate the sum:

$$A = \overset{3}{\left[\left(3+2i\right)\right]^2} + \overset{4}{\left[\left(5-6i\right)\right]^3}$$

As we know, we get:

$$A = \overset{3}{\left[\left(3+2i\right)\right]^2} + \overset{4}{\left[\left(5-6i\right)\right]^3} = \overset{3^2}{\left(3+2i\right)^2} + \overset{4^3}{\left(5-6i\right)^3} = \overset{9}{\left(3+2i\right)^2} + \overset{64}{(5-6i)^3} = \ldots$$

6. Calculate the sum:

$$A = \sqrt[3]{\overset{4+5i}{\left(3-2i\right)}} + \sqrt[3]{\overset{4-5i}{\left(5+6i\right)}}$$

As we know, we get:

$$A=\sqrt[3]{\left(3-2i\right)^{4+5i}}+\sqrt[3]{(5+6i)^{4-5i}}=\sqrt[3]{3-2i}^{\sqrt[3]{4+5i}}+\sqrt[3]{5+6i}^{\sqrt[3]{4-5i}}=$$

$$=\left(\sqrt[3]{3-2i}+\sqrt[3]{5+6i}\right)^{\sqrt[3]{4+5i}\cdot\sqrt[3]{4-5i}}=\left(\sqrt[3]{3-2i}+\sqrt[3]{5+6i}\right)^{\sqrt[3]{4^2+5^2}}=$$

$$=\left(\sqrt[3]{3-2i}+\sqrt[3]{5+6i}\right)^{\sqrt[3]{41}}$$

16. GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS OF DEGREE

Consider the complex number of degree:

$$z_g=\left(x_b+iy_b\right)^{x_h+iy_h} \qquad (16.1)$$

where:

$$z_b=x_b+iy_b$$

is the base and:

$$z_h=x_h+iy_h$$

is the hyperthetis.

i.e.:

$$z_g=z_b^{z_h}=\left(x_b+iy_b\right)^{x_b+iy_h}$$

Consider figure 5, a 4-dimensional system of orthogonal coordinates (x_b, y_b, x_h, y_h) .

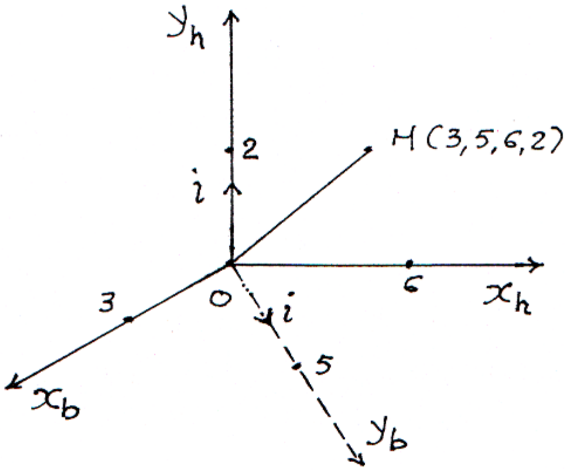


fig. 5

$y'_h y_h$ and $y'_b y_b$ are the imaginary axes, while $x'_b x_b$ and $x'_h x_h$ are the real axes. So, a complex number of degree, e.g.,

$$z_g=\left(3+5i\right)^{6+2i} \qquad (16.2)$$

for which: $x_b = 3$, $y_b = 5$, $x_h = 6$ and $y_h = 2$, is represented on the (x_b, y_b, x_h, y_h) orthogonal coordinate system as the point **M** with coordinates, **(3, 5, 6, 2).**

So, while the hitherto known complex numbers are represented on a plane (2-dimensional space), complex numbers of degree are represented – by contrast – in 4-dimensional space, i.e., in the orthogonal coordinate system (x_b, y_b, x_h, y_h) **.**

The modulus of a complex number of degree, $\left|z_g\right|$**, is:**

$$\left|z_g\right| = \sqrt{x_b^2 + y_b^2 + x_h^2 + y_h^2}$$

and represents the length $(OM) = \left|z_g\right|$ **(figure 5).**

17. COMPLEX FUNCTIONS OF DEGREE TRANSFORMATION OF COMPLEX FUNCTIONS OF DEGREE

The notation:

$$w_g = f_g(z_g) \quad (17.1)$$

represents a complex function of degree, where:

$$w_g = w_b^{w_h}$$

$$(w_b = u_b + iv_b \text{ and } w_h = u_h + iv_h)$$

Also:

$$z_g = z_b^{z_h}$$

$$(z_b = x_b + iy_b \text{ and } z_h = x_h + iy_h)$$

So, equation (17.1), constitutes a transformation on a complex function of degree, with the condition that:

$$u_b = u_b(x_b, y_b)$$

$$v_b = v_b(x_b, y_b)$$

$$u_h = u_h(x_h, y_h)$$

$$v_h = v_h(x_h, y_h)$$

Consider the transformation of degree:

$$w_g = \overset{2}{3}z_g \overset{2}{+6} \qquad (17.2)$$

which can also be written as:

$$\overset{w_h}{w_b} = \overset{2}{3}\cdot \overset{z_h}{z_b} + \overset{2}{6} \Leftrightarrow$$

$$\overset{u_h+iv_h}{\left(u_b + iu_b\right)} = \overset{2}{3}\overset{x_h+iy_h}{\left(x_b + iy_b\right)} + \overset{2}{6} \Leftrightarrow$$

$$\overset{u_h+iv_h}{\left(u_b + iu_b\right)} = \overset{2(x_h+iy_h)\cdot 2}{\left[3(x_b + iy_b) + 6\right]} \Leftrightarrow$$

$$\overset{u_h+iv_h}{\left(u_b + iv_b\right)} = \overset{4x_h+i4y_h}{\left(3x_b + i3y_b + 6\right)} \quad (17.3)$$

From (17.3) we get:

$$\left.\begin{aligned} u_h + iv_h &= 4x_h + i4y_h \\ u_b + iv_b &= 3x_b + 6 + i3y_b \end{aligned}\right\} \quad (17.4)$$

a) From the first of these equations, (17.4), we get:

$$\left.\begin{aligned} u_h &= 4x_h \\ v_h &= 4y_h \end{aligned}\right\} \quad (17.5)$$

Consider now two orthogonal frames of reference, (x_h, y_h) and (u_h, v_h), figure 6:

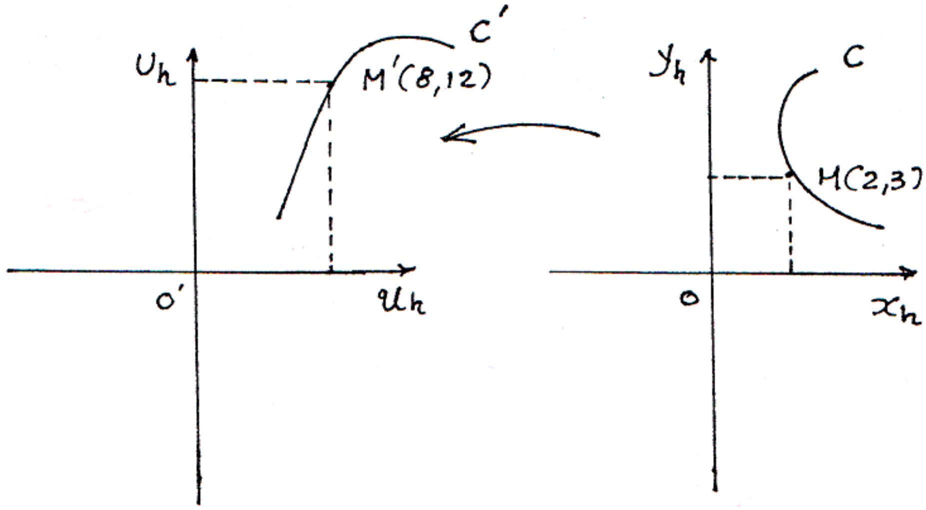


fig. 6

A point, e.g. *M (2,3)* in the coordinate system, (x_h, y_h) - depending on (17.5) - has an image, M', in the coordinate system, (u_h, v_h), with coordinates:

$$u_h = 4 \cdot x_h = 4 \cdot 2 = 8$$
$$v_h = 4 \cdot y_h = 4 \cdot 3 = 12$$

i.e., the point M' (8,12), and in general, a set of points, *C***,** in the coordinate system (x_h, y_h) has a corresponding image, C', in the coordinate system (u_h, v_h).

b) From the second of these equations, (17.4), we get:

$$\left.\begin{aligned} u_b &= 3x_b + 6 \\ v_b &= 3y_b \end{aligned}\right\} \qquad (17.6)$$

Consider now two orthogonal frames of reference, (x_h, y_h) and (u_h, v_h), figure 7:

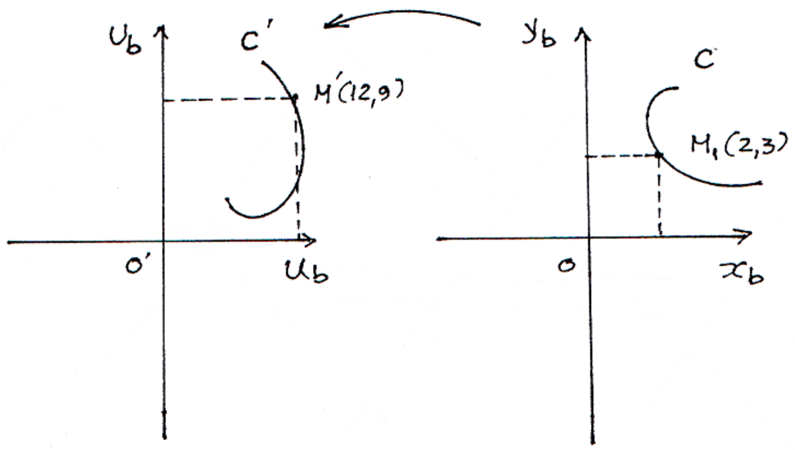


fig. 7

A point, e.g. *M (2,3)* in the coordinate system, (x_h, y_h) - depending on (17.5) - has an image, M', in the coordinate system, (u_h, v_h), with coordinates:

$$\begin{aligned} u_b &= 3 \cdot 2 + 6 = 12 \\ v_b &= 3 \cdot 3 = 9 \end{aligned}$$

i.e., the point M' (12,9), and in general, a set of points, *C***,** in the coordinate system (x_h, y_h) has a corresponding image, C', in the coordinate system (u_h, v_h) - depending of course on equations (17.6)

So, the transformation of degree (17.2) represents two transformations, figures 6 & 7, which shall be called the conjugate transformations of the given transformation.

18. COMPLEX NUMBERS OF DEGREE, SECOND TYPE (I_2)

Definition: we define complex numbers Z_G of degree (second type, I_2) as complex numbers of degree of the form:

$$\boxed{Z_G = \overset{\kappa}{\alpha} + \overset{\lambda}{\beta} \cdot \overset{i}{i}} \quad (18.1)$$

where $\overset{\kappa}{\alpha}$ is the real part and $\overset{\lambda}{\beta}$ is the imaginary part.

Complex numbers Z_G of degree (second type, I_2), undergo the four basic mathematical operations (addition, subtraction, multiplication, division) of category $\overset{1}{\mathbf{C}}$ and $\overset{0}{\mathbf{C}}$.

Definition: We define as the additive of the complex number, $Z_{G,+}$, of a complex number Z_G of degree (second type, I_2) (18.1) as the number:

$$Z_{G,+} = (\alpha + \kappa) + (\beta + \lambda)(i + i) \Rightarrow$$

$$\boxed{Z_{G,+} = (\alpha + \kappa) + 2(\beta + \lambda)i} \quad (18.5)$$

Example: If we have:

$$Z_G = \overset{6}{4} + \overset{2}{5} \cdot \overset{i}{i} \quad (18.6)$$

then $Z_{G,+}$, is:

$$\begin{aligned} Z_{G,+} &= (4+6) + (5+2)(i+i) \Rightarrow \\ Z_{G,+} &= 10 + 7 \cdot 2i \Rightarrow \qquad (18.7) \\ Z_{G,+} &= 10 + 14i \end{aligned}$$

The number $Z_{G,+}$ of (18.7) is represented on the plane $E_{G,+}$ of the additive complex numbers $Z_{G,+}$ by the point M(10,14), figure 7(a).

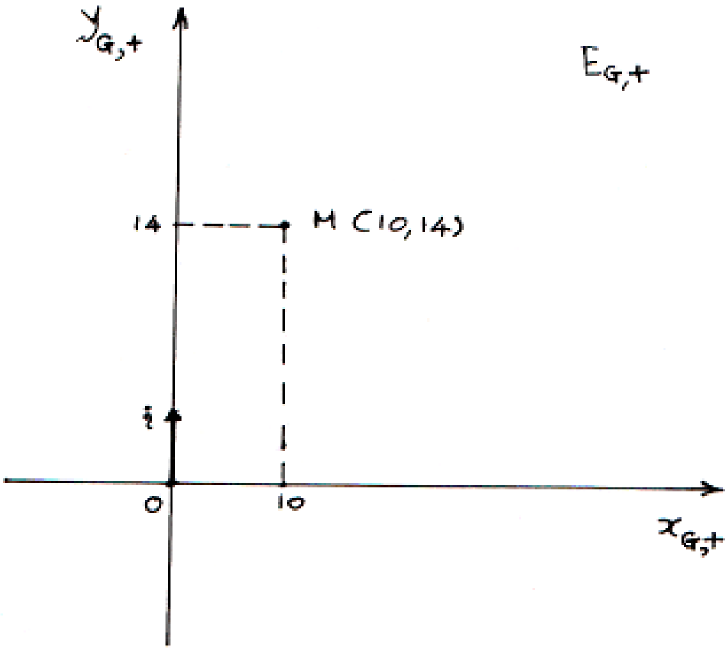


fig. 7(a)

PART TWO
AXIOMATIC FOUNDATION OF REAL NUMBERS OF DEGREE, CATEGORY $\overset{0}{\mathbf{C}}$

1. BASIC OPERATIONS ON NUMBERS OF DEGREE, CATEGORY $\overset{0}{\mathbf{C}}$

1. Addition

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 = \overset{\kappa+\lambda}{\left(a_1 + a_2\right)}$$

2. Subtraction

$$\overset{\kappa}{a}_1 - \overset{\lambda}{a}_2 = \overset{\kappa+\lambda}{\left(a_1 - a_2\right)}$$

3. Multiplication

$$\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2 = \overset{\kappa+\lambda}{\left(a_1 \cdot a_2\right)}$$

4. Division

$$\overset{\kappa}{a}_1 : \overset{\lambda}{a}_2 = \overset{\kappa-\lambda}{\left(a_1 : a_2\right)}$$

Based on the above, it can be shown that:

If $\overset{\kappa}{a}_1$, $\overset{\lambda}{a}_2$, $\overset{\mu}{a}_3$ are elements of the set $\mathbf{S}_{\mathbf{g}}'$ of real numbers of degree, category $\overset{0}{\mathbf{C}}$, then:

a. $\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2$ and $\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2$ belong to $\mathbf{S}_{\mathbf{g}}'$ (closure property).

b. $\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 = \overset{\lambda}{a}_2 + \overset{\kappa}{a}_1$ (commutative property).

c. $\overset{\kappa}{a}_1 + (\overset{\lambda}{a}_2 + \overset{\mu}{a}_3) = (\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2) + \overset{\mu}{a}_3$ (associative property).

d. $\overset{\kappa}{a}_1 \cdot (\overset{\lambda}{a}_2 \cdot \overset{\mu}{a}_3) = (\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2) \cdot \overset{\mu}{a}_3$ (associative property).

e. $\overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2 = \overset{\lambda}{a}_2 \cdot \overset{\kappa}{a}_1$ (commutative property).

f. $\overset{\kappa}{a}_1 \cdot (\overset{\lambda}{a}_2 + \overset{\mu}{a}_3) = \overset{\kappa}{a}_1 \cdot \overset{\lambda+\mu}{(a_2 + a_3)}$ and not $\overset{\kappa}{a}_1 \cdot (\overset{\lambda}{a}_2 + \overset{\mu}{a}_3) = \overset{\kappa}{a}_1 \cdot \overset{\lambda}{a}_2 + \overset{\kappa}{a}_1 \cdot \overset{\mu}{a}_3$ (non-distributive property).

g. $\overset{\kappa}{a}_1 + \overset{0}{0} = \overset{0}{0} + \overset{\kappa}{a}_1 = \overset{\kappa}{a}_1$.

$\overset{0}{0}$ is called the identity element of addition.

h. $\overset{\kappa}{a}_1 \cdot \overset{0}{1} = \overset{0}{1} \cdot \overset{\kappa}{a}_1 = \overset{\kappa}{a}_1$.

$\overset{0}{1}$ is called the identity element of multiplication.

i. For every real number of degree $\overset{\kappa}{a}_1$ there is only one real number of degree $\overset{\rho}{a}$ of $\mathbf{S}_{\mathbf{g}}'$, such that:

$$\overset{\kappa}{a}_1 + \overset{\rho}{a} = \overset{0}{0}.$$

$\overset{\rho}{a}$ is called the additive inverse of $\overset{\kappa}{a}_1$ and is denoted by:

$$\overset{\rho}{a} = -\overset{-\kappa}{a}_1$$

j. For every $\overset{\kappa}{\alpha}_1 \neq \overset{0}{0}$, there is only one real number of degree $\overset{\rho}{a}$ of $\mathbf{S}_{\mathbf{g}}'$, such that:

$$\overset{\kappa}{a}_1 \cdot \overset{\rho}{a} = \overset{\rho}{a} \cdot \overset{\kappa}{a}_1 = \overset{0}{1}.$$

$\overset{\rho}{a}$ is called the multiplicative inverse of $\overset{\kappa}{a}_1$ and is denoted by:

$$\overset{\rho}{\alpha} = \left(\frac{1}{a_1}\right)^{\overset{-\kappa}{}}$$

k. If:

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 = \overset{\mu}{a}_3 + \overset{\nu}{a}_4$$

then:

$$\overset{\kappa}{a}_1 + \overset{\lambda}{a}_2 - \overset{-\mu}{a}_3 = \overset{\nu}{a}_4$$

$$\overset{\kappa}{a}_1 = \overset{\mu}{a}_3 + \overset{\nu}{a}_4 - \overset{\lambda}{a}_2$$

I.e., in an equality with real numbers of degree in category $\overset{0}{\mathbf{C}}$, when a term is moved from one side of the equation to the other, then, the sign of the base and the hyperthetis changes, something that does not happen in category $\overset{1}{\mathbf{C}}$ that we dealt with previously.

2. IDENTITIES AND DEFINITIONS

Identity 1: If $\overset{\kappa}{a}$ is a real number of degree, then we have:

$$\overset{\kappa}{a} = \overset{0}{\alpha} \cdot \overset{\kappa}{1} = \alpha \cdot \overset{\kappa}{1}$$

Identity 2: According to the known, we have:

$$\sqrt{\overset{\kappa}{a}} = \sqrt{\overset{\kappa/2}{a}}$$

Identity 3:

$$\sqrt[\nu]{\overset{\kappa}{a}} = \sqrt[\nu]{\overset{\kappa/\nu}{a}}$$

Identity 4: Two real numbers of degree $\overset{\kappa}{a}$ and $\overset{\lambda}{\beta}$ are equal,

$$\overset{\kappa}{a} = \overset{\lambda}{\beta}$$

if and only if:

$$\left.\begin{array}{r} a = \beta \\ \text{and } \kappa = \lambda \end{array}\right\}$$

Identity 5:

$$\alpha^{\overset{\mu}{\nu}} = \alpha^{\overset{\mu\cdot\nu}{\nu}}$$

Identity 6: In category $\overset{0}{\mathbf{C}}$, by definition:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} = \underbrace{\overset{\kappa+\mu}{a} \cdot \overset{\kappa+\mu}{a} \cdot \overset{\kappa+\mu}{a} \cdots \overset{\kappa+\mu}{a}}_{\nu-times} = \overset{\nu(\kappa+\mu)}{a^{\nu}}$$

Identity 7: We have:

$$\overset{\kappa}{a} + \overset{\kappa}{a} = \left(\overset{\kappa+\kappa}{a+a}\right) = \left(\overset{2\kappa}{2a}\right) \text{ and not}$$

$$\overset{\kappa}{a} + \overset{\kappa}{a} = 2\overset{\kappa}{a}$$

Identity 8: If:

$$a = a_1 + a_2 + a_3 + \dots + a_\nu$$

then by definition:

$$\overset{\kappa}{a} = (a_1 + a_2 + \overset{\kappa}{a_3} + \ldots + a_\nu)$$

EXAMPLES

Calculate, in the category $\overset{0}{\mathbf{C}}$:

1. The sum:

$$A = \overset{3}{2} + \overset{5}{3} - \overset{1,5}{5} + \overset{4}{2} - \overset{0}{1}$$

As we know, we get:

$$A = \overset{3}{2} + \overset{5}{3} - \overset{1,5}{5} + \overset{4}{2} - \overset{0}{1} = \overset{3+5+(1,5)+4+0}{(2 + 3 - 5 + 2 - 1)} = \overset{10,5}{1}$$

2. The sum:

$$A = \overset{-3}{-4} + \overset{-4}{8} - \overset{5}{2}$$

We get:

$$A = \overset{-3}{-4} + \overset{-4}{8} - \overset{5}{2} = \overset{-3+(-4)+5}{(-4 + 8 - 2)} = \overset{-2}{2}$$

3. The product:

$$A = \overset{5}{2} \cdot \overset{-6}{3} \cdot \overset{-1}{4} \cdot \overset{0}{6}$$

We get:

$$A = \overset{5}{2} \cdot \overset{-6}{3} \cdot \overset{-1}{4} \cdot \overset{0}{6} = \overset{5+(-6)+(-1)+0}{(2 \cdot 3 \cdot 4 \cdot 6)} = \overset{-2}{144}$$

4. The product:

$$A = \overset{-1,2}{-1,8} \cdot \overset{-4}{3,5} \cdot \overset{-2,8}{6,7}$$

will be:

$$A = \overset{-1,2}{-1,8} \cdot \overset{-4}{3,5} \cdot \overset{-2,8}{6,7} = \overset{-1,2+(-4)+(-2,8)}{(-1,8 \cdot 3,5 \cdot 6,7)} = \overset{-8}{42,21}$$

5. The ratio:

$$A = \frac{\overset{-2}{8}}{\overset{4}{4}}$$

will be:

$$A = \frac{\overset{-2}{8}}{\overset{4}{4}} = \overset{-2-4}{\left(\frac{8}{4}\right)} = \overset{-6}{2}$$

6. The ratio:

$$A = \frac{\overset{-1}{4}}{\overset{-8}{2}}$$

will be:

$$A = \frac{\overset{-1}{4}}{\overset{-8}{2}} = \overset{-1-(-8)}{\left(\frac{4}{2}\right)} = \overset{7}{2}$$

7. The function:

$$A = \frac{\overset{3}{2} + \overset{-1}{3} - \overset{0}{2} \cdot \overset{2}{3} + \sqrt{\overset{6}{4}} + \sqrt{\overset{2}{0}}}{(\overset{4}{2} + \overset{-6}{1}) \cdot (\overset{3}{3} + \overset{4}{5}) + \sqrt{\overset{1}{4}} + \sqrt{\overset{0}{9}}}$$

As we know, we get:

$$A = \frac{\overset{3}{2} + \overset{-1}{3} - \overset{0}{2} \cdot \overset{2}{3} + \sqrt{\overset{6}{4}} + \sqrt{\overset{2}{0}}}{(\overset{4}{2} + \overset{-6}{1}) \cdot (\overset{3}{3} + \overset{4}{5}) + \sqrt{\overset{1}{4}} + \sqrt{\overset{0}{9}}} =$$

$$\frac{\overset{3}{2} + \overset{-1}{3} - \overset{2}{6} + \overset{3}{2} + \overset{1}{0}}{\overset{-2}{3} \cdot \overset{7}{8} + \overset{0,5}{2} + \overset{0}{3}} = \frac{\overset{3+(-1)+2+3+1}{\left(2+3-6+2+0\right)}}{\overset{-2+7+0,5+0}{\left(24+2+3\right)}} =$$

$$= \frac{\overset{8}{1}}{\overset{5,5}{29}} = \overset{8-5,5}{\left(\frac{1}{29}\right)} = \overset{2,5}{\left(\frac{1}{29}\right)}.$$

8. The function:

$$A=\frac{\left(\overset{3}{3}\right)^{2}+\left(\overset{2}{4}\right)^{3}+\sqrt[3]{\overset{9}{8}}-\overset{3}{2^{2}}\cdot\overset{2}{4^{2}}\cdot\left(\overset{2}{4}\right)^{2}}{\left(\overset{4}{2}\right)^{3}+\left(\overset{3}{5}\right)^{3}+\sqrt[5]{\overset{25}{32}}+\overset{4}{4^{3}}\cdot\overset{1}{3^{1}}\cdot\left(\overset{3}{2}\right)^{3}}$$

is:

$$A=\frac{\overset{2\cdot3}{\left(3^{2}\right)}+\overset{3\cdot2}{\left(4^{3}\right)}+\overset{3}{2}-\overset{3\cdot2}{(2^{2})}\cdot\overset{2\cdot2}{\left(4^{2}\right)}\overset{2\cdot2}{\left(4^{2}\right)}}{\overset{3\cdot4}{\left(2^{3}\right)}+\overset{3\cdot3}{\left(5^{3}\right)}+\overset{5}{2}+\overset{4\cdot3}{\left(4^{3}\right)}\overset{1\cdot1}{\left(3^{1}\right)}\overset{3\cdot3}{\left(2^{3}\right)}}=$$

$$=\frac{\overset{6}{9}+\overset{6}{64}+\overset{3}{2}-\overset{6}{4}\cdot\overset{4}{16}\cdot\overset{4}{16}}{\overset{12}{8}+\overset{9}{125}+\overset{5}{2}+\overset{12}{64}\cdot\overset{1}{3}\cdot\overset{9}{8}}=\frac{\overset{6+6+3}{\left(9+64+2\right)}-\overset{6+4+4}{\left(4\cdot16\cdot16\right)}}{\overset{12+9+5}{\left(8+125+2\right)}+\overset{12+1+9}{\left(64\cdot3\cdot8\right)}}=$$

$$=\frac{\overset{15}{75}-\overset{16}{256}}{\overset{26}{135}+\overset{22}{1536}}=\frac{\overset{15+16}{\left(75-256\right)}}{\overset{26+22}{\left(135+1536\right)}}=\frac{\overset{31}{181}}{\overset{48}{1671}}=\overset{31-48}{\left(\frac{181}{1671}\right)}=$$

$$=\overset{-17}{\left(\frac{181}{1671}\right)}\approx\overset{-17}{0{,}10}$$

9. Prove identity 3, i.e.:

$$\left(\overset{\kappa}{\alpha}\right)^{\nu}=\overset{\nu\cdot\kappa}{\left(\alpha^{\nu}\right)}$$

Proof:

$$\left(\overset{\kappa}{a}\right)^{\nu}=\underbrace{\overset{\kappa+0}{a}\cdot\overset{\kappa+0}{a}\cdot\overset{\kappa+0}{a}\cdots\overset{\kappa+0}{a}}_{\nu-times}=$$

$$=\overset{\overbrace{\kappa+\kappa+\kappa+\ldots+\kappa}^{\nu-times}}{a^{\nu}}=\overset{\nu\cdot\kappa}{(\alpha^{\nu})}\qquad\text{QED.}$$

10. Prove identity 5, i.e.:

$$\overset{\mu}{\alpha^{\nu}}=\overset{\mu\cdot\nu}{\alpha^{\nu}}$$

Proof:

$$\overset{\mu}{\alpha}^{\nu} = \underbrace{\overset{\mu+0}{a}\cdot\overset{\mu+0}{a}\cdot\overset{\mu+0}{a}\cdots\overset{\mu+0}{a}}_{\nu-times} == \overset{\overbrace{\mu+\mu+\mu+...+\mu}^{\nu-times}}{(a^{\nu})} = \overset{\nu\cdot\mu}{(\alpha^{\nu})} \quad \text{QED.}$$

11. Because:

$$\overset{\kappa}{a}+\overset{\lambda}{\beta} = \overset{\kappa+\lambda}{\left(a+\beta\right)}$$

then:

$$\overset{\kappa}{a} = \overset{\kappa+\lambda}{\left(a+\beta\right)} - \overset{-\lambda}{\beta}$$

according the known identity.

Indeed,

$$\overset{\kappa}{a} = \overset{\kappa+\lambda}{\left(a+\beta\right)} - \overset{-\lambda}{\beta} = \overset{\kappa+\lambda-\lambda}{\left(a+\beta-\beta\right)} = \overset{\kappa}{a} \qquad \text{QED.}$$

12. Because:

$$\overset{\kappa}{a}\cdot\overset{\lambda}{\beta} = \overset{\kappa+\lambda}{\left(a\cdot\beta\right)}$$

then:

a) $\overset{\kappa}{a} = \dfrac{\overset{\kappa+\lambda}{\left(a\cdot\beta\right)}}{\overset{\lambda}{\beta}}$ and **b)** $\overset{\kappa}{a}\cdot\overset{\lambda}{\beta} - \overset{-(\kappa+\lambda)}{\left(a\cdot\beta\right)} = \overset{0}{0}$

Indeed,

a) $\overset{\kappa}{a} = \dfrac{\overset{\kappa+\lambda}{\left(a\cdot\beta\right)}}{\overset{\lambda}{\beta}} = \overset{\kappa+\lambda-\lambda}{\left(\dfrac{a\cdot\beta}{\beta}\right)} = \overset{\kappa}{a}$ QED.

b) $\overset{\kappa}{a}\cdot\overset{\lambda}{\beta} - \overset{(\kappa+\lambda)}{\left(a\cdot\beta\right)} = \overset{\kappa}{a}\cdot\overset{\lambda}{\beta} - \overset{-(\kappa+\lambda)}{\left(a\cdot\beta\right)} = \overset{\kappa+\lambda-(\kappa+\lambda)}{(\alpha\beta-\alpha\beta)} = \overset{0}{0}$ QED.

3. GENERAL EXAMPLES OF THE CATEGORY $\overset{0}{C}$ OF MATHEMATICS

1. Calculate the sum:

$$A = \overset{6-3i}{(3+2i)} + \overset{-3-3i}{(8-2i)} - \overset{2i}{(6+3i)}$$

As we know, we get:

$$A = \overset{6-3i}{\left(3+2i\right)} + \overset{-3-3i}{\left(8-2i\right)} - \overset{2i}{(6+3i)} = \overset{6-3i+(-3-3i)+2i}{\left(3+2i+8-2i-6-3i\right)} = \overset{3-4i}{\left(5-3i\right)}$$

2. Calculate the product:

$$A = \overset{1+i}{(2+3i)} \cdot \overset{1-i}{(2-3i)} \cdot \overset{3+5i}{(4+6i)} \cdot \overset{3-5i}{\left(4-6i\right)}$$

As we know, we get:

$$A = \overset{1+i}{(2+3i)} \cdot \overset{1-i}{(2-3i)} \cdot \overset{3+5i}{(4+6i)} \cdot \overset{3-5i}{\left(4-6i\right)} = \overset{1+i+1-i+3+5i+3-5i}{\left[\left(2^2+3^2\right)\left(4^2+6^2\right)\right]} = \overset{8}{676}$$

3. Calculate the ratio:

$$A = \frac{\overset{i}{\left(28i\right)}}{\overset{i}{(3i)}}$$

becomes:

$$A = \overset{i-i}{\left(\frac{28i}{3i}\right)} = \overset{0}{\left(\frac{28}{3}\right)} = \frac{28}{3}$$

4. Because:

$$\overset{\kappa+\lambda i}{\left(\alpha+\beta i\right)} + \overset{\mu+\nu i}{\left(\gamma+\delta i\right)} = \overset{\kappa+\mu+(\lambda+\nu)i}{[\alpha+\gamma+(\beta+\delta)i]}$$

then:

$$\overset{\kappa+\lambda i}{\left(\alpha+\beta i\right)} = \overset{\kappa+\mu+(\lambda+\nu)i}{[\alpha+\gamma+(\beta+\delta)i]} - \overset{-(\mu+\nu i)}{\left(\gamma+\delta i\right)}$$

Indeed,

$$\overset{\kappa+\lambda i}{\left(\alpha+\beta i\right)} = \overset{\kappa+\mu+(\lambda+\nu)i-(\mu+\nu i)}{\left[\alpha+\gamma+(\beta+\delta)i-\left(\gamma+\delta i\right)\right]} =$$

$$\overset{\kappa+\mu+\lambda i+\nu i-\mu-\nu i}{[\alpha+\gamma+\beta i+\delta i-\gamma-\delta i]} = \overset{\kappa+\lambda i}{\left(\alpha+\beta i\right)} \quad \text{QED.}$$

5. **Calculate the sum:**

$$A = \eta\mu^{2}x^{\sigma\upsilon\nu^{2}x} + \sigma\upsilon\nu^{2}x^{\eta\mu^{2}x}$$

As we know, we get:

$$A = \eta\mu^{2}x^{\sigma\upsilon\nu^{2}x} + \sigma\upsilon\nu^{2}x^{\eta\mu^{2}x} = \left(\eta\mu^{2}x + \sigma\upsilon\nu^{2}x\right)^{\sigma\upsilon\nu^{2}x+\eta\mu^{2}x} = 1^{1}$$

6. Calculate the value of the function:

$$A = \eta\mu^{2}x^{e^{ix}} + \sigma\upsilon\nu^{2}x^{e^{-ix}}$$

for $x = \dfrac{\pi}{2}$

As we know, we get:

$$A = \eta\mu^{2}x^{e^{ix}} + \sigma\upsilon\nu^{2}x^{e^{-ix}} = \left(\eta\mu^{2}x + \sigma\upsilon\nu^{2}x\right)^{e^{ix}+e^{-ix}} = 1^{\sigma\upsilon\nu x + i\eta\mu x + \sigma\upsilon\nu x - i\eta\mu x} = 1^{2\cdot 0} = 1^{0} = 1 = 1$$

7. Calculate the function:

$$A = \frac{\left[(2+i)^{1-i}\right]^{2} + \left[(4-i)^{3+i}\right]^{3} + \left((6+3i)^{2}\right)^{3i}}{\left(\sqrt{(3+5i)^{2}}\right)^{4i} + \left(\sqrt{(5-6i)^{5}}\right)^{20i}}$$

As we know, we get:

$$A = \frac{\left[(2+i)^{2}\right]^{2(1-i)} + \left[(4-i)^{3}\right]^{3(3+i)} + \left((6+3i)^{2}\right)^{3i\cdot 2}}{(3+5i)^{4i:2} + (5-6i)^{20i:5}} =$$

$$= \frac{(4-1+4i)^{2-2i} + (64-48i-12-i)^{9+3i} + (36-9+36i)^{6i}}{(3+5i)^{2i} + (5-6i)^{4i}} =$$

$$= \frac{(4-1+4i+64-48i-12-i+36-9+36i)^{2-2i+9+3i}}{(3+5i+5-6i)^{2i+4i}} =$$

$$= \frac{(82-9i)^{11+i}}{(8-i)^{6i}} = \left(\frac{82-9i}{8-i}\right)^{11+i/6i}$$

8. Solve the equation of degree:

$$\overset{x^2}{(x^2)}\overset{-4x}{-5}\cdot\overset{0}{x}+\overset{3}{4}=\overset{0}{0} \quad (3.1)$$

SOLUTION

The given equation of degree (3.1) in category $\overset{0}{C}$ becomes:

$$\overset{x^2}{\left(x^2\right)}-\overset{-4x}{5}\cdot\overset{0}{x}+\overset{3}{4}=\overset{0}{0} \Leftrightarrow$$

$$\overset{x^2}{\left(x^2\right)}-\overset{-4x+0}{\left(5\cdot x\right)}+\overset{3}{4}=\overset{0}{0} \Leftrightarrow$$

$$\overset{x^2-4x+3}{\left(x^2-5x+4\right)}=\overset{0}{0} \Leftrightarrow$$

$$\overset{x^2-4x+3}{\left(x^2-5x+4\right)}=\overset{0}{0} \qquad (3.2)$$

whose conjugate equations are:

$$\left.\begin{array}{l} x^2-4x+3=0 \\ x^2-5x+4=0 \end{array}\right\} \quad (3.3)$$

The hyperthetis equation yields the roots:

$$\left(x_{h_1}=3 \ \ and \ \ x_{h_2}=1\right) \qquad (3.4)$$

And the base equation yields the roots:

$$\left(x_{b_1}=4 \ \ and \ \ x_{b_2}=1\right) \qquad (3.5)$$

The roots (3.4) and (3.5) are the required roots of the function of degree (3.1), in category $\overset{0}{C}$ of mathematics.

Obviously, the equation of degree (3.1) would yield different roots if solved in category $\overset{1}{C}$ of mathematics.

NOTE: As mentioned before, any equation of degree, solved in the category $\overset{0}{C}$ of mathematics, should be set equal to $\overset{0}{0}$ in order to find its roots.

4. NOTEWORTHY EQUATIONS OF DEGREE, CATEGORY $\overset{0}{C}$

1. Based on the aforementioned:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} = a^{\overset{\nu(\kappa+\mu)}{\nu}}$$

from which:

$$\left(\overset{\kappa}{a}\right)^{\nu} = a^{\overset{\kappa\nu}{\nu}} \qquad \text{and}$$

$$(\alpha)^{\overset{\mu}{\nu}} = a^{\overset{\mu\nu}{\nu}}$$

2. Prove that:

$$\left[\left(\overset{\kappa}{a}\right)^{\overset{\tau}{\rho}}\right]^{\overset{\mu}{\nu}} = a^{\overset{(\kappa+\tau)\rho\nu+\mu\nu}{\rho\nu}}$$

Proof: From (1), we get:

$$\left(\overset{\kappa}{a}\right)^{\overset{\tau}{\rho}} = a^{\overset{(\kappa+\tau)\rho}{\nu}} \text{, therefore:}$$

$$\left[\left(\overset{\kappa}{a}\right)^{\overset{\tau}{\rho}}\right]^{\overset{\mu}{\nu}} = \left(a^{\overset{(\kappa+\tau)\rho}{\rho}}\right)^{\overset{\mu}{\nu}} = a^{\overset{(\kappa+\tau)\rho\nu+\mu\nu}{\rho\nu}} \text{,} \qquad \text{QED.}$$

3. Prove that:

$$\sqrt[\overset{\mu}{\nu}]{\overset{\kappa}{a}} = \sqrt[\nu]{\overset{\frac{\kappa-\mu\nu}{\nu}}{a}}$$

Proof: we set:

$$\sqrt[\overset{\mu}{\nu}]{\overset{\kappa}{a}} = \overset{B}{A}$$

so we have:

$$\overset{\kappa}{a} = \left(\overset{B}{A}\right)^{\overset{\mu}{\nu}} \quad \text{or} \quad \overset{\kappa}{a} = \mathrm{A}^{\overset{(\mathrm{B}+\mu)^{\nu}}{\nu}}$$

from which we get:

$$\left.\begin{array}{l}(\mathrm{B}+\mu)\nu=\kappa \\ \text{and } A^{\nu}=a\end{array}\right\}$$

which are rearranged to give:

$$\left.\begin{array}{l}B=\dfrac{\kappa-\mu\nu}{\nu} \\ A=\sqrt[\nu]{a}\end{array}\right\} \text{QED.}$$

4. Prove that:

$$\sqrt[\rho]{\tau}{\sqrt[\nu]{\mu}{\sqrt{\kappa}{a}}}$$

$$\sqrt[\rho]{\vphantom{x}}^{\tau}\!\!\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\kappa}\, a = \sqrt[\rho\nu]{a}^{\,\frac{\kappa-\mu\nu-\tau\rho\nu}{\rho\nu}}$$

Proof: According to the equation in (3) above, we have:

$$\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\kappa}\, a = \sqrt[\nu]{a}^{\,\frac{\kappa-\mu\nu}{\nu}}, \text{ therefore:}$$

$$\sqrt[\rho]{\vphantom{x}}^{\tau}\!\!\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\kappa}\, a = \sqrt[\rho]{\vphantom{x}}^{\tau}\!\!\sqrt{\sqrt[\nu]{a}^{\,\frac{\kappa-\mu\nu}{\nu}}} = \sqrt[\rho\nu]{a}^{\,\frac{\kappa-\mu\nu-\tau\rho\nu}{\rho\nu}} \text{, QED.}$$

5. Prove that:

$$\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\kappa}\, a \cdot \sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\lambda}\, \beta = \sqrt[\nu]{a\cdot\beta}^{\,\frac{(\kappa-\mu\nu)+(\lambda-\mu\nu)}{\nu}}$$

Proof: We have:

$$\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\kappa}\, a = \sqrt[\nu]{a}^{\,\frac{\kappa-\mu\nu}{\nu}} \text{, and}$$

$$\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\lambda}\, \beta = \sqrt[\nu]{\beta}^{\,\frac{\lambda-\mu\nu}{\nu}}$$

Multiplying the parts we get:

$$\sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\kappa}\, a \cdot \sqrt[\nu]{\vphantom{x}}^{\mu}\!\!\sqrt{\vphantom{x}}^{\lambda}\, \beta = \sqrt[\nu]{a}^{\,\frac{\kappa-\mu\nu}{\nu}} \cdot \sqrt[\nu]{\beta}^{\,\frac{\lambda-\mu\nu}{\nu}} = \sqrt[\nu]{a\cdot\beta}^{\,\frac{(\kappa-\mu\nu)+(\lambda-\mu\nu)}{\nu}} \quad \text{QED}$$

6. Prove that:

$$\sqrt[\substack{\mu\\ \nu}]{\substack{\kappa\\ a}} : \sqrt[\substack{\mu\\ \nu}]{\substack{\lambda\\ \beta}} = \sqrt[\nu]{\overset{\frac{\kappa-\lambda}{\nu}}{\dfrac{a}{\beta}}}$$

Proof: Exactly as previous example

7. Prove that:

$$\sqrt[\substack{\mu\\ \nu}]{\left(\substack{\kappa\\ a}\right)^{\substack{\tau\\ \rho}}} = \overset{\frac{(\kappa+\tau)\rho-\mu\nu}{\nu}}{\sqrt[\nu]{a^{\rho}}}$$

Proof: As we know, we have

$$\left(\substack{\kappa\\ a}\right)^{\substack{\tau\\ \rho}} = \overset{(\kappa+\tau)\rho}{a^{\rho}} ,$$

Therefore, again from what we know:

$$\sqrt[\substack{\mu\\ \nu}]{\overset{(\kappa+\tau)\rho}{a^{\rho}}} = \overset{\frac{(\kappa+\tau)\rho-\mu\nu}{\nu}}{\sqrt[\nu]{a^{\rho}}} \text{ , QED.}$$

8. Prove that:

$$\underbrace{\left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} \cdot \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} \cdot \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} \textbf{.....} \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} = \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu\cdot\rho}}}_{\rho-times}$$

Proof: As we know, the above can be written as:

$$\underbrace{\overset{(\kappa+\mu)\nu}{a^{\nu}} \cdot \overset{(\kappa+\mu)\nu}{a^{\nu}} \cdot \overset{(\kappa+\mu)\nu}{a^{\nu}} \cdots\cdot \overset{(\kappa+\mu)\nu}{a^{\nu}}}_{\rho-times} = \overset{(\kappa+\mu)\rho\nu}{a^{\rho\nu}} = \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \rho\nu}} \text{ , QED.}$$

9. Prove that:

$$\underbrace{\left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} + \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} + \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} \textbf{.....} \left(\substack{\kappa\\ a}\right)^{\substack{\mu\\ \nu}} = \rho \overset{(\kappa+\mu)\rho\nu}{a^{\nu}}}_{\rho-times}$$

Proof: As we know, the above can be written as:

$$\underbrace{\overset{(\kappa+\mu)\nu}{a^{\nu}} + \overset{(\kappa+\mu)\nu}{a^{\nu}} + \overset{(\kappa+\mu)\nu}{a^{\nu}} \cdots\cdot \overset{(\kappa+\mu)\nu}{a^{\nu}}}_{\rho-times} = \rho\overset{(\kappa+\mu)\rho\nu}{a^{\nu}} \text{ , QED.}$$

10. It can also be shown that:

$$\text{a) } \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} + \left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa+\mu)\nu+(\lambda+\tau)\rho}{\left(a^{\nu}+\beta^{\rho}\right)}$$

$$\text{b) } \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} - \left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa+\mu)\nu+(\lambda+\tau)\rho}{\left(a^{\nu}-\beta^{\rho}\right)}$$

$$\text{c) } \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} \cdot \left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa+\mu)\nu+(\lambda+\tau)\rho}{\left(a^{\nu}\cdot\beta^{\rho}\right)}$$

$$\text{d) } \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}} : \left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}} = \overset{(\kappa+\mu)\nu-(\lambda+\tau)\rho}{\left(a^{\nu}:\beta^{\rho}\right)}$$

The above proofs are very easy, based on what has been mentioned so far.

11. If: $\left(\overset{\kappa}{a}\right)^{\overset{y}{x}} = \overset{B_1}{A_1}$, then by definition:

$$\log_{\overset{\kappa}{a}} \overset{\mathrm{B}_1}{\mathrm{A}_1} = \overset{y}{x} \qquad \text{(B)}$$

Because we have:

$$\left(\overset{\kappa}{a}\right)^{\overset{y}{x}} = \overset{(\kappa+y)x}{a^{x}}$$

then from these relationships we get:

$$\overset{(\kappa+y)x}{a^{x}} = \overset{B_1}{A_1} \quad \text{or} \quad \left.\begin{aligned}(\kappa+y)x &= B_1\\ \alpha^{x} &= A_1\end{aligned}\right\}\text{(c)}$$

Solving the simultaneous equations **(c)** for **x** and **y** we find the value of the logarithm (B).

12. $\overset{\kappa}{a}, \left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right), \left[\left(\overset{\kappa}{a}+\overset{\lambda}{\beta}\right)+\overset{\lambda}{\beta}\right], \dots\dots\dots = \overset{\kappa}{a}, \overset{\kappa+\lambda}{(a+\beta)}, \overset{\kappa+2\lambda}{(a+2\beta)}, \dots\dots\dots\dots \overset{\kappa+(\nu-1)\lambda}{(a+(\nu-1)\beta)}$

is an arithmetic progression of degree (ν terms), with first term, $\overset{\kappa}{a}$, and common difference, $\overset{\lambda}{\beta}$.

13. For the above arithmetic progression, prove that:

a) The νth term of the sequence is:

$$\tau = \overset{\kappa+(\nu-1)\lambda}{(a+(\nu-1)\beta)}$$

b) The sum of ν terms is:

$$\Sigma = \left(\overset{\nu\kappa+\lambda\left(\frac{\nu-1}{2}\right)^{\nu}}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right)$$

Proof: a) By definition, the νth term of an arithmetic progression is:

$$\tau = \overset{\kappa+(\nu-1)\lambda}{(a+(\nu-1)\beta)}$$

b) To find the sum of ν terms we proceed as follows:

$$\Sigma = \overset{\kappa}{a} + \overset{\kappa+\lambda}{(a+\beta)} + \overset{\kappa+2\lambda}{(a+2\beta)} + \dots\dots\dots\dots + \overset{\kappa+(\nu-1)\lambda}{(a+(\nu-1)\beta)}$$

$$\text{or } \Sigma = \left(\overset{\nu\kappa+\lambda(1+2+3+\dots(\nu-1))}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right)$$

but:

$$\mathbf{1+2+3+\dots+(\nu-1)} = \left(\frac{\nu-1}{2}\right)\nu$$

which gives us:

$$\Sigma = \left(\overset{\nu\kappa+\lambda\left(\frac{\nu-1}{2}\right)\nu}{\frac{[2a+(\nu-1)\beta]\nu}{2}} \right), \text{ QED}$$

14. $\overset{\kappa}{a}, \left(\overset{\kappa}{a}\cdot\overset{\lambda}{\beta}\right), \left[(\overset{\kappa}{a}\cdot\overset{\lambda}{\beta})\cdot\overset{\lambda}{\beta}\right], \dots\dots\dots = \overset{\kappa}{a}, \overset{\kappa+\lambda}{(a\cdot\beta)}, \overset{\kappa+2\lambda}{(a\cdot\beta^2)}, \dots\dots\dots\dots \overset{\kappa+(\nu-1)\lambda}{(a\cdot\beta^{(\nu-1)})}$,

is a geometric progression of degree (ν terms), with first term, $\overset{\kappa}{a}$, and common ratio, $\overset{\lambda}{\beta}$.

a) The νth term of the sequence is:

$$\tau = \overset{\kappa+(\nu-1)\lambda}{(a\cdot\beta^{\nu-1})}$$

b) The sum of ν terms is:

$$\Sigma = \left[\frac{a(\beta^{\nu}-1)}{\beta-1}\right]^{\nu\kappa+\lambda\left(\frac{\nu-1}{2}\right)\nu}$$

These are proven in the manner demonstrated for the arithmetic progression.

15. Other sequences of degree:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}},\ \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}+\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}},\ \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}+\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}}+\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}},\ \ldots.$$

arithmetic progression with first term, $\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}$, and common difference $\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}}$.

Also:

$$\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}},\ \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}\cdot\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}},\ \left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}\cdot\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}}\cdot\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}}\ \ldots.$$

geometric progression with first term, $\left(\overset{\kappa}{a}\right)^{\overset{\mu}{\nu}}$, and common ratio $\left(\overset{\lambda}{\beta}\right)^{\overset{\tau}{\rho}}$.

NOTE: All the above mathematical formulae, can be reduced to the equivalent formulae of the hitherto known mathematics by setting the hyperthetes of the numbers of degree equal to 0.

5.TOPOLOGY IN MATHEMATICS OF DEGREE
BASIC TOPOLOGICAL OPERATIONS

In the mathematics of degree, the following definition holds:

Definition: To any number of degree:

$$\overset{n}{S}$$

where S > 0 and n is a natural number,

corresponds a topological surface, surface area S and genus n. Obviously, the inverse definition also holds:

To any topological surface, surface area S and genus n, corresponds a number of degree $\overset{n}{S}$.

For example, based on the above definition, the number $\overset{0}{4}$, represents a sphere (as we know, a sphere has genus **n = 0**) with surface area 4. Similarly, a torus (donut), surface area 10 (as we know, a torus has genus **n = 1**), represents the number of degree $\overset{1}{10}$.

Note: In the above definition, the form (shape) is irrelevant, as long as the surface area is S and genus n.

As we will shortly see, in the Topology of mathematics of degree, we can perform operations (add, subtract, multiply, divide and so on) on different shapes (topological spaces) - in category $\overset{1}{C}$ or $\overset{0}{C}$ - something that was not possible hitherto.

EXAMPLES

1a) In category $\overset{1}{C}$, figure 8, what shape results from the addition of a sphere with surface area S_1, a cube with surface area S_2 and a torus (donut) with surface area S_3 ?

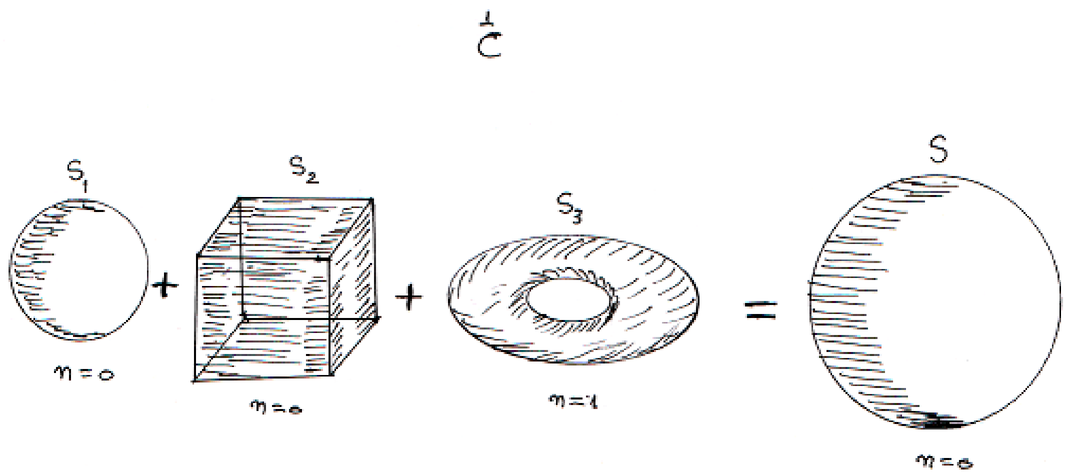


fig. 8

SOLUTION

The given translates into:

$$\overset{0}{S}_1+\overset{0}{S}_2+\overset{1}{S}_3=\overset{0\cdot 0\cdot 1}{\left(S_1+S_2+S_3\right)}=\overset{0}{\left(S_1+S_2+S_3\right)} \qquad (1)$$

i.e., the result is a sphere with surface area S=$S_1+S_2+S_3$, figure 8.

b) What shape results in category $\overset{0}{C}$?

SOLUTION

The given translates into:

$$\overset{0}{S_1}+\overset{0}{S_2}+\overset{1}{S_3}=\overset{0+0+1}{\left(S_1+S_2+S_3\right)}=\overset{1}{\left(S_1+S_2+S_3\right)} \qquad (2)$$

i.e., the result is a torus (donut) with surface area S=$S_1+S_2+S_3$, figure 9.

$\overset{0}{C}$

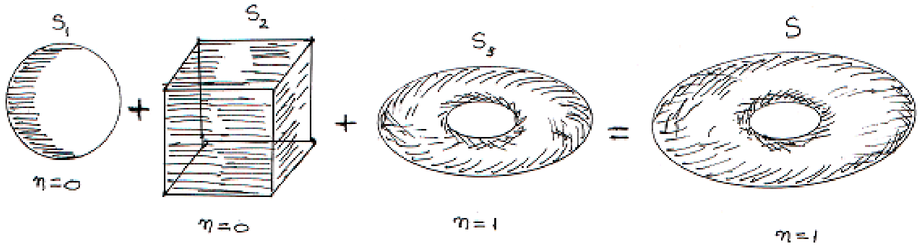


fig. 9

2. In figure 10, the topological surfaces A, B, C, D have surface area S_1, S_2, S_3, S_4 respectively.

Find, in category $\overset{1}{C}$, the shape that results from the following operations:

$\overset{1}{C}$

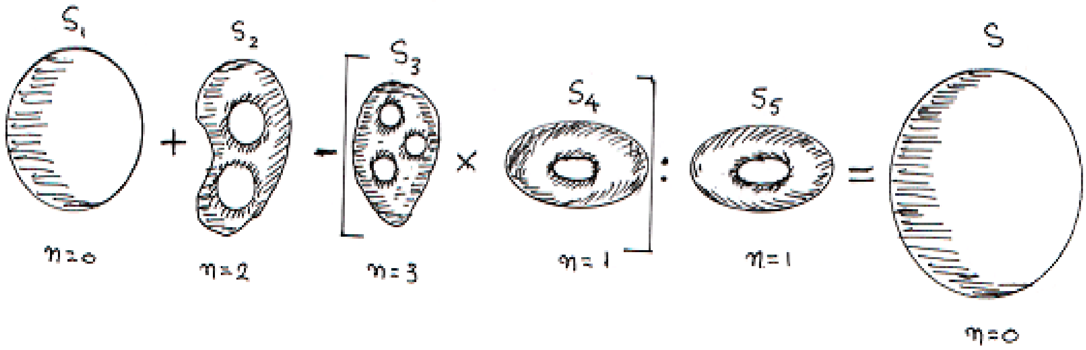


fig. 10

SOLUTION

a. In category $\overset{1}{C}$, the given problem gives:

$$\overset{0}{S_1} + \overset{2}{S_2} - (\overset{3}{S_3} \times \overset{1}{S_4}) : \overset{1}{S_5} = (S_1 \overset{0\cdot 2}{+} S_2) - (S_3 \overset{3\cdot 1}{\cdot} S_4) : \overset{1}{S_5} \Rightarrow$$

$$(S_1 \overset{0}{+} S_2) - (S_3 \overset{3}{\cdot} S_4) : \overset{1}{S_5} \Rightarrow$$

$$(S_1 \overset{0}{+} S_2) - \overset{3:1}{\left(\frac{S_3 \cdot S_4}{S_5}\right)} \Rightarrow$$

$$(S_1 \overset{0}{+} S_2) - \overset{3}{\left(\frac{S_3 \cdot S_4}{S_5}\right)} \Rightarrow$$

$$\left(S_1 + S_2 \overset{0\cdot 3}{-} \frac{S_3 \cdot S_4}{S_5}\right) \Rightarrow$$

$$\left(S_1 + S_2 \overset{0}{-} \frac{S_3 \cdot S_4}{S_5}\right) = \overset{0}{S} \qquad (3)$$

i.e., the result is a sphere with surface area $S = S_1 + S_2 - \dfrac{S_3 \cdot S_4}{S_5}$

3. In category $\overset{0}{C}$, what shape results if we multiply v tori (donuts), each with surface area $\dfrac{1}{E}$, E > 0?

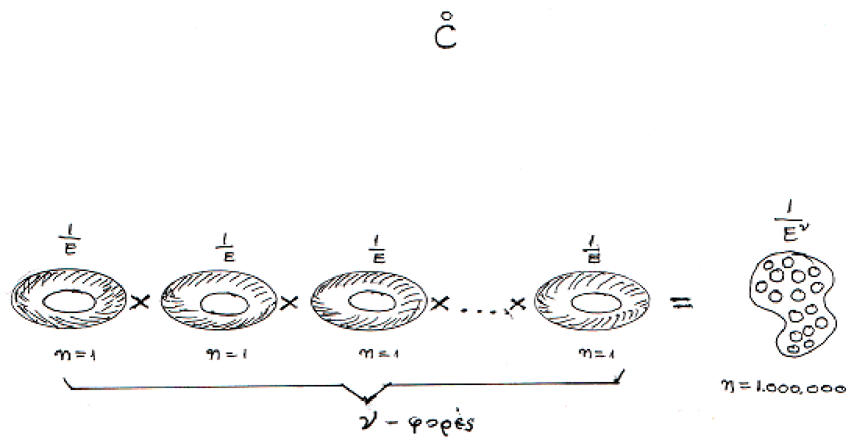


fig. 11

SOLUTION

As we know, in category $\overset{0}{\mathbf{C}}$ we have:

$$\overset{1}{\left(\frac{1}{E}\right)}_1 \cdot \overset{1}{\left(\frac{1}{E}\right)}_2 \cdot \overset{1}{\left(\frac{1}{E}\right)}_3 \cdot \ldots \cdot \overset{1}{\left(\frac{1}{E}\right)}_\nu = \overset{1+1+1+1+\ldots+1}{\left(\frac{1}{E^\nu}\right)} = \overset{\nu}{\left(\frac{1}{E^\nu}\right)} \qquad (4)$$

Therefore, from (4), e.g. for ν = 1,000,000 tori (donuts), when multiplied the result is a topological surface with surface area $\frac{1}{E^\nu}$ (very small surface area) and genus n = ν = 1.000.000, **i.e., a «sponge» surface, with vary small surface area and many «handles».**

4. (The inverse problem)

In category $\overset{1}{\mathbf{C}}$, what topological surface is A;

$$\mathbf{A} = \frac{\overset{10}{\mathbf{2}} + \overset{2}{\mathbf{3}} + \overset{1}{\mathbf{4}} \cdot \overset{2}{\mathbf{5}}}{\overset{20}{\mathbf{5}}} \qquad (5)$$

SOLUTION

From (5), we get:

$$\mathbf{A} = \frac{\overset{10}{\mathbf{2}} + \overset{2}{\mathbf{3}} + \overset{1}{\mathbf{4}} \cdot \overset{2}{\mathbf{5}}}{\overset{20}{\mathbf{5}}} = \frac{\overset{10 \cdot 2 \cdot 2}{(\mathbf{2} + \mathbf{3} + \mathbf{20})}}{\overset{20}{\mathbf{5}}} = \frac{\overset{40}{\mathbf{25}}}{\overset{20}{\mathbf{5}}} = \overset{40:20}{\left(\frac{\mathbf{25}}{\mathbf{5}}\right)} \Leftrightarrow$$

$$\mathbf{A} = \overset{2}{\mathbf{5}} \qquad (6)$$

Therefore, the required topological surface has surface area **S = 5** and genus **n = 2**, figure 12.

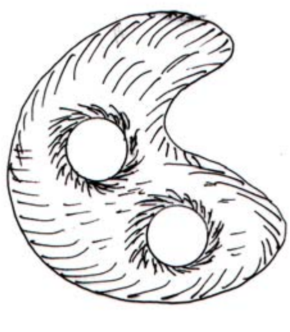


fig. 12

5. Other examples (for the reader).

- **a.** In category $\overset{1}{C}$ and $\overset{0}{C}$, what surface results from the multiplication of a sphere, surface area S_1 and a torus (donut) surface area S_2 ?
- **b.** In category $\overset{1}{C}$ and $\overset{0}{C}$, what surface results from the addition of three tori (donuts), each of surface area S_1 ?

SUMMARY OF CATEGORY $\overset{0}{C}$ OF MATHEMATICS

As we have seen, the categories $\overset{0}{C}$ and $\overset{1}{C}$ of mathematics, are two categories with different axiomatic foundation.

To avoid unnecessary repetition, it is enough to emphasise here that:

The same reasoning and methods applied to various mathematical problems of category $\overset{1}{C}$ (e.g. functions, equations, integrals, differential equations, complex numbers and so on) considered in the above chapters, can also be applied to solve the same problems in category $\overset{0}{C}$.

We must also emphasise that, in the mathematics of degree, any given problem can have mora than one solution. **Namely, it has a different solution (usually though not always) in category $\overset{1}{C}$ and a different solution in category $\overset{0}{C}$.**

This is obviously not the case for the hitherto known mathematics, as they have one and only one axiomatic foundation.

Finally, after all that has been mentioned, we arrive at the following conclusion:

The whole Science of mathematics can be founded on two distinctive categories, namely:

- **a. Category $\overset{1}{C}$, and**
- **b. Category $\overset{0}{C}$**

and none other.

6. TRANSFORMATION OF THE HITHERTO KNOWN MATHEMATICS TO MATHEMATICS OF DEGREE
METHOD OF INDUCTION

In the preceding chapters we established the axiomatic foundation of mathematics, in categories $\overset{1}{\mathbf{C}}$ and $\overset{0}{\mathbf{C}}$, and demonstrated the methods and reasoning that are applied to the mathematics of degree.

Having said all this, the following conclusion arises:
If we take any mathematical relationship (e.g. equation, function, integral etc) of the hitherto known mathematics and convert its digits (variables or constants) into numbers of degree of the form $\overset{\kappa}{a}$ (i.e. assign hyperthetes), we get the equivalent mathematical relationship (e.g. equation, function, integral etc) of the mathematics of degree.
Choice of the hyperthetes can be arbitrary or not.

For example, the equation,

$$\mathbf{ax^2 + bx + c = 0} \qquad (1)$$

transformed into the mathematics of degree can take the form (one of many that can be chosen):

$$\overset{3}{\mathbf{a}} \cdot \mathbf{x}^2 + \overset{-1}{\mathbf{b}} \cdot \overset{3}{\mathbf{x}} + \overset{5}{\mathbf{c}} = \overset{1}{\mathbf{0}} \qquad (2)$$

Furthermore, solving the equation (2), in category $\overset{0}{\mathbf{C}}$ and $\overset{1}{\mathbf{C}}$ of mathematics, we can examine and compare any conclusions that might arise with the respective conclusions arising from the familiar solutions of equation (1).

The same can also be performed, for example, on the following integral from our hitherto known mathematics:

$$\int \sqrt{\mathbf{e^x + 4}} \cdot \mathbf{dx} \qquad (3)$$

The integral (3), transformed into the mathematics of degree - in one of its possible forms - becomes:

$$\int \sqrt[\overset{3}{2}]{\overset{2}{\mathbf{e}}^{\mathbf{x}} + \overset{-5}{\mathbf{4}}}\, \mathbf{dx} \qquad (4)$$

Furthermore, solving the integral (4), in category $\overset{0}{\mathbf{C}}$ and $\overset{1}{\mathbf{C}}$ of mathematics, we can examine and compare any conclusions that might arise with the respective conclusions arising from the familiar solutions of equation (3).

The same method demonstrated in the above two examples can applied to any mathematical relationship (e.g. equation, function, integral and so on) of the hitherto known mathematics.

Definition: The method of transforming any relationship A of the hitherto known mathematics into a relationship of the mathematics of degree - by transforming each digit to a number of degree (i.e., assign different hyperthetes on top of each digit of relationship A) – shall be called the «Method of induction» of the mathematics of degree.

Consequently, we can immediately conclude from the above that the field of study of the mathematics of degree is immense in the science of mathematics.

NOTEWORTHY OBSERVATION

Based on the «Method of induction», any problem of the hitherto known mathematics (e,g. Fermat's last theorem, the Riemann conjecture, etc...), can be transformed into the mathematics of degree where it can be studied and proven, if in fact a solution exists.

THE FUNDAMENTAL THEOREMS OF THE MATHEMATICS OF DEGREE

1. A mentioned above, the four basic operations (addition, subtraction, multiplication, division), in categories $\overset{1}{C}$ and $\overset{0}{C}$, are the same regarding the «base» but differ regarding the «hyperthetes».

Consequently, we can state the following theorem:
THEOREM: (Theorem of common solutions).
In the mathematics of degree, the solution of any mathematical problem, in categories $\overset{1}{C}$ and $\overset{0}{C}$, is always common with regard to the «base» and different (not always) with regard to the «hyperthetis»

2. Additionally, as mentioned previously, the mathematics of degree are founded in two and only two categories of axiomatic foundation, i.e., category $\overset{1}{C}$ and category $\overset{0}{C}$.

Consequently, we can state the following theorem:
THEOREM: (Theorem of complete solution).
The solution of any mathematical problem is complete, if and only if, the problem is solved in both categories of axiomatic foundation, i.e., category $\overset{1}{C}$ and category $\overset{0}{C}$.

3. In the hitherto known mathematics, because the axiomatic foundation is one and only one, the solution of a mathematical problem is one and only one.

By contrast, the same does not occur in the mathematics of degree because there are two axiomatic foundations, namely, that of category $\overset{1}{C}$ and that of category $\overset{0}{C}$.
Consequently, we can state the following theorem:

THEOREM: (Theorem of uncertainty of solutions).
Given a mathematical problem, we cannot proceed to its correct solution, if we do not know in advance (a priori), in which category or categories ($\overset{1}{C},\overset{0}{C}$) the solution is required.

As we can see, the three above theorems have deep implications regarding the «purely mathematical» aspect of mathematics, but even more so in the philosophy of mathematics.

EPILOGUE

Having mentioned all this, we have outlined the basic principles and way of thinking applied to the mathematics of degree, in the axiomatic foundation of categories $\overset{0}{C}$ and $\overset{1}{C}$.

But, the field of the mathematics of degree is a huge field of mathematical study.

So, further research into the mathematics of degree is certain to yield many more notable theorems, conclusions, identities and so on, as well as numerous philosophical conclusions that affect the philosophy of mathematics.

The mathematics of degree is a completely new branch of mathematics, and as such is still on its first «steps».

Time will judge the qualitative and quantitative contribution of the mathematics of degree to the science of mathematics.

Χρήστος Τσόλκας
June 2007

SUPPLEMENT
ANALYTIC GEOMETRY OF DEGREE
BASIC CONCEPTS

A. Graphical representation of numbers of degree

As is known, a number of degree **A**, is expressed by the following notation:

$$\mathbf{A} = \overset{\mathbf{k}}{\mathbf{a}} \qquad (1)$$

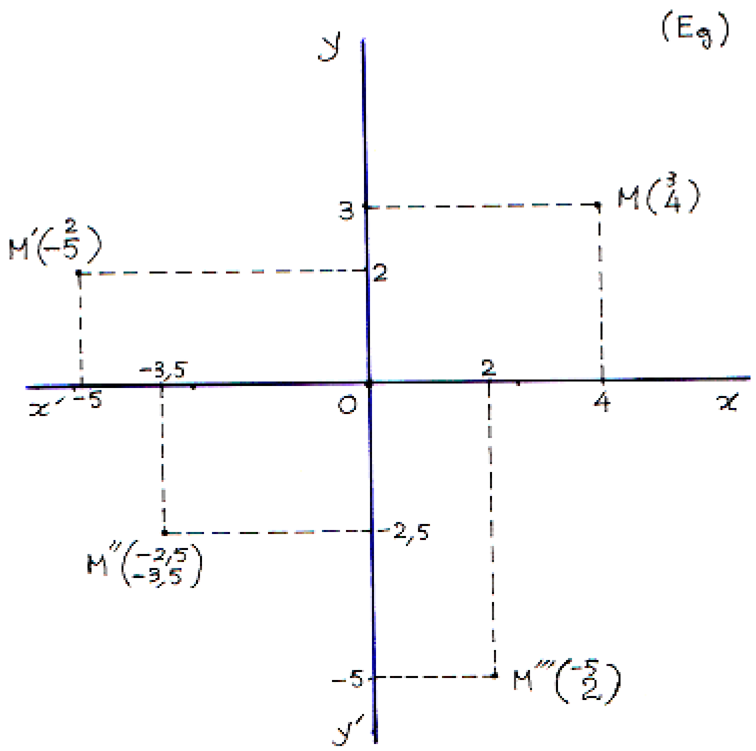
where, a is the base and k is the hyperthetis (degree).

For example, in the number of degree:

$$\mathbf{A} = \overset{\mathbf{3}}{\mathbf{4}} \qquad (2)$$

a = 4 and **k = 3**.

Additionally, as mentioned in previous chapters, on a plane $\mathbf{E_g}$ in an orthogonal coordinate system **x-y**, a number of degree, for example $\overset{\mathbf{3}}{\mathbf{4}}$, is the point $\mathbf{M}\left(\overset{\mathbf{3}}{\mathbf{4}}\right)$, which has coordinates **x = 4** and **y = 3**, fig(a):



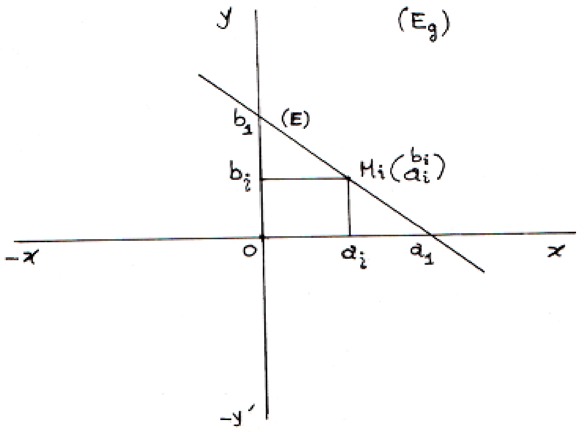
fig(a)

Consequently:

a. If for example we have the equation:

$$\frac{\mathbf{x}}{\mathbf{a_1}} + \frac{\mathbf{y}}{\mathbf{b_1}} = \mathbf{1} \qquad (3)$$

then, equation (3) on the plane $\mathbf{E_g}$ represents a straight line (E), fig(1):



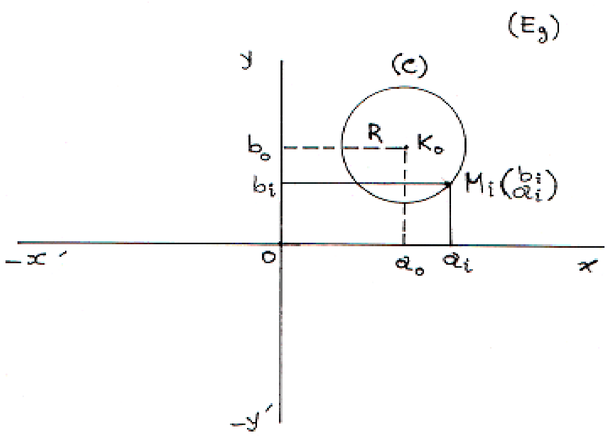
fig(1)

We will call the numbers of degree $\mathbf{M_i}\begin{pmatrix}\mathbf{b_i}\\ \mathbf{a_i}\end{pmatrix}$ that lie on the line (E) **«linear numbers of degree»**.

b. If for example we have the equation:

$$(\mathbf{x}-\mathbf{a_0})^2 + (\mathbf{y}-\mathbf{b_0})^2 = \mathbf{R}^2 \qquad (4)$$

then, equation (4) on the plane $\mathbf{E_g}$ represents the circumference of a circle (**C**) radius **R** with center $\mathbf{K_0}$ having coordinates $\mathbf{K_0}(\mathbf{a_0},\mathbf{b_0})$, fig(2):



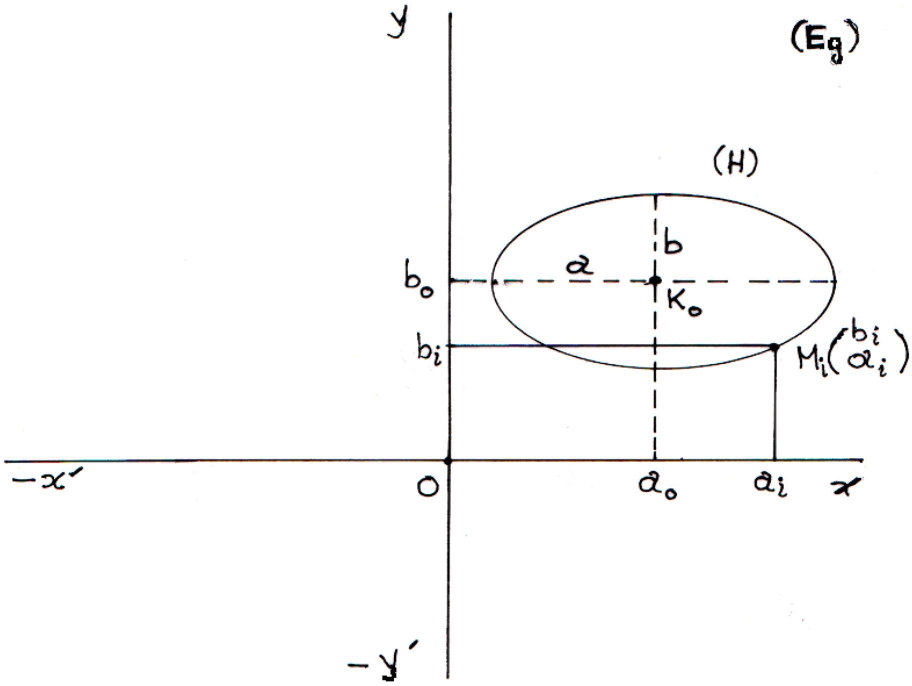
fig(2)

We will call the numbers of degree $\mathbf{M}_{\mathbf{i}}\begin{pmatrix}\mathbf{b}_{\mathbf{i}}\\ \mathbf{a}_{\mathbf{i}}\end{pmatrix}$ that lie on the circumference (C) **«circular numbers of degree»**.

c. If for example we have the equation:

$$\frac{(\mathbf{x}-\mathbf{a}_0)^2}{\mathbf{a}^2}+\frac{(\mathbf{y}-\mathbf{b}_0)^2}{\mathbf{b}^2}=\mathbf{1} \qquad (5)$$

then, equation (5) on the plane $\mathbf{E_g}$ represents an ellipse (**H**) with major and minor axis **a** and **b** respectively and center $\mathbf{K_0}$ having coordinates $\mathbf{K_0}(\mathbf{a_0},\mathbf{b_0})$, fig(3):



fig(3)

We will call the numbers of degree $\mathbf{M}_{\mathbf{i}}\begin{pmatrix}\mathbf{b}_{\mathbf{i}}\\ \mathbf{a}_{\mathbf{i}}\end{pmatrix}$ that lie on the ellipse (H) **«elliptical numbers of degree»**.

Similarly, in the manner mentioned above, we can have **«parabolic numbers of degree»**, **«hyperbolic numbers of degree»** and so on, for any equation $\mathbf{f(x,y)=0}$ from our familiar analytic geometry.

B. Graphical representation of equations of degree

Let us suppose for example that we have the following equation of degree (category $\overset{1}{\mathbf{C}}$):

$$\mathbf{f_g(x,y) = \overset{1}{0}} \qquad (6)$$

of which the **simultaneous equations** are:

$$\mathbf{f_b(x,y) = 0} \qquad (7)$$

the **base equation**, and

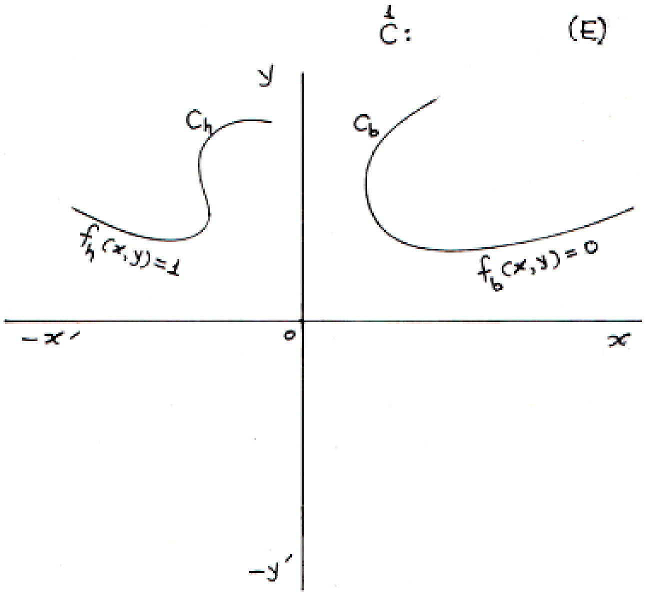
$$\mathbf{f_h(x,y) = 1} \qquad (8)$$

the **hyperthetis (degree) equation**
where x and y are real numbers.

Also, as is known,

$$\mathbf{f_g(x,y) = \frac{f_h(x,y)}{f_b(x,y)}} \qquad (9)$$

Now, lets take a plane (E) in an orthogonal coordinate system **x-y**, fig(4):



fig(4)

Then, in the case of equation (9), the base equation (7) is represented by the curve $\mathbf{C_b}$ and the hyperthetis equation (8) is represented by a different curve $\mathbf{C_h}$.

As we can see, our initial equation of degree (6) is represented by two curves on a plane (E) in an orthogonal coordinate system x-y, namely, the base curve C_b and the hyperthetis curve C_h.

By contrast, in the familiar Analytic Geometry, on a plane (E) in an orthogonal coordinate system x-y, an equation:

$$\mathbf{f(x,y)=0}$$

is always represented by a single curve C (a line, circle, ellipse etc).

This is the basic difference between the familiar Analytic Geometry and the Analytic Geometry of Degree.

Let us mention now a few examples of Analytic Geometry of Degree.

EXAMPLES

Example 1: Given the equation of degree (category $\overset{1}{\mathbf{C}}$),

$$\overset{y}{\mathbf{x}} + \overset{x}{\mathbf{y}} = \overset{1}{\mathbf{0}} \qquad (13)$$

represent it graphically on a plane (E) in an orthogonal coordinate system **x-y**.

Solution:

Equation (13) can be rearranged to give:

$$\overset{y}{\mathbf{x}} + \overset{x}{\mathbf{y}} = \left(\mathbf{x} \overset{x\cdot y}{+} \mathbf{y}\right), \text{ so:}$$

$$\left(\mathbf{x} \overset{x\cdot y}{+} \mathbf{y}\right) = \overset{1}{\mathbf{0}} \qquad (14)$$

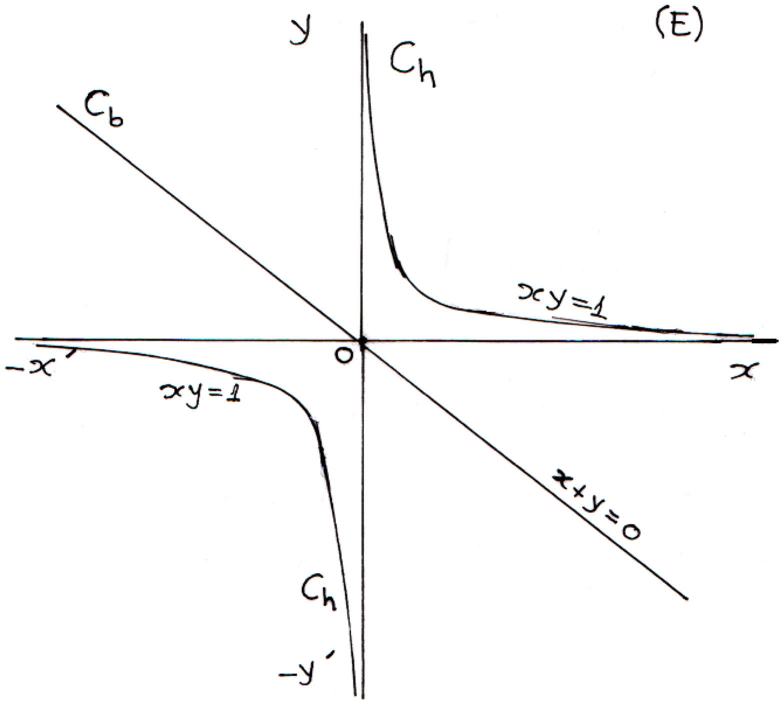
In the equation of degree (14) the base equation is:

$$\mathbf{x + y = 0} \qquad (15)$$

and the hyperthetis equation:

$$\mathbf{xy = 1} \qquad (16)$$

Consequently, the graphical representation of the equation of degree (13), on a plane (E) in an orthogonal coordinate system **x-y**, is the line $\mathbf{x + y = 0}$ given by equation (15) and the hyperbola $\mathbf{xy = 1}$ given by equation (16), fig(5):



fig(5)

Example 2: Given the equation of degree (category $\overset{\mathbf{0}}{\mathbf{C}}$),

$$\overset{\mathbf{2x+1}}{\mathbf{x}} - \overset{\mathbf{3y+2}}{\mathbf{y}} = \overset{\mathbf{0}}{\mathbf{0}} \qquad (17)$$

represent it graphically on a plane (E) in an orthogonal coordinate system **x-y**.

Solution:

Equation (17) can be rearranged to give:

$$\overset{\mathbf{2x+1}}{\mathbf{x}} - \overset{\mathbf{3y+2}}{\mathbf{y}} = \overset{\mathbf{(2x+1)+(3y+2)}}{(\mathbf{x}-\mathbf{y})} = \overset{\mathbf{2x+3y+3}}{(\mathbf{x}-\mathbf{y})}, \text{ so:}$$

$$\overset{\mathbf{2x+3y+3}}{(\mathbf{x}-\mathbf{y})} = \overset{\mathbf{0}}{\mathbf{0}} \qquad (18)$$

In the equation of degree (18) the base equation is:

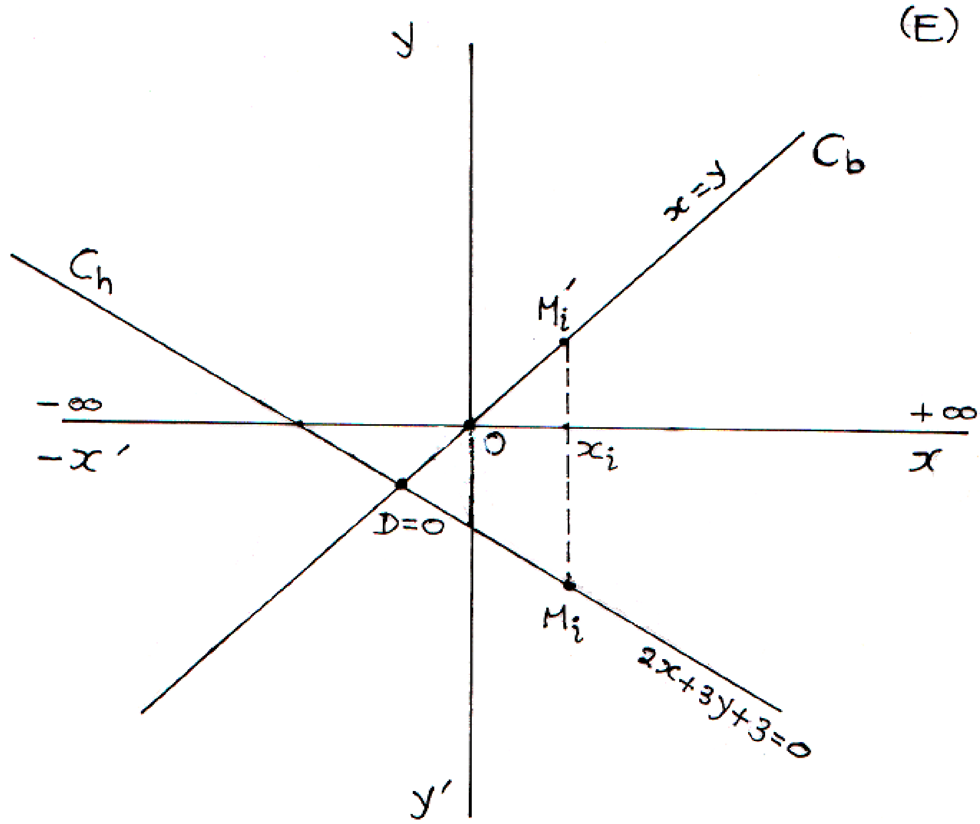
$$\mathbf{x - y = 0} \qquad (19)$$

and the hyperthetis equation:

$$\mathbf{2x + 3y + 3 = 0} \qquad (20)$$

Consequently, the graphical representation of the equation of degree (17), on a plane (E) in an orthogonal coordinate system **x-y**, is the line **x - y = 0** given by equation (19) and the line **2x + 3y + 3 = 0** given by equation (20), fig(6):

Note: What is the graphical representation of the equation of degree (17) in the category $\overset{1}{\mathbf{C}}$? (Exercise for the reader)..



fig(6)

Example 3: Given the equation of degree (category $\overset{1}{\mathbf{C}}$),

$$\overset{x}{\mathbf{0}}+\overset{y}{\mathbf{0}}+\overset{1}{\mathbf{0}}+\overset{2}{\mathbf{x}}-\left(\overset{2}{\mathbf{4}}\right)^{\overset{4}{3}}=\overset{1}{\mathbf{0}} \qquad (21)$$

represent it graphically on a plane (E) in an orthogonal coordinate system **x-y**.

Solution:

Equation (21) can be rearranged to give:

$$\overset{x}{\mathbf{0}}+\overset{y}{\mathbf{0}}+\overset{1}{\mathbf{0}}+\overset{2}{\mathbf{x}}-\left(\overset{2}{\mathbf{4}}\right)^{\overset{4}{3}}=\overset{x}{\mathbf{0}}+\overset{y}{\mathbf{0}}+\overset{1}{\mathbf{0}}+\overset{2}{\mathbf{x}}-\overset{(2\cdot 4)^3}{\left(\mathbf{4}^3\right)}=$$

$$\overset{x}{\mathbf{0}}+\overset{y}{\mathbf{0}}+\overset{1}{\mathbf{0}}+\overset{2}{\mathbf{x}}-\overset{512}{\mathbf{64}}=\overset{x\cdot y\cdot 1\cdot 2\cdot 512}{(\mathbf{0}+\mathbf{0}+\mathbf{0}+\mathbf{x}-\mathbf{64})}=$$

$$\overset{1024xy}{(\mathbf{x}-\mathbf{64})}, \text{ so:}$$

$$\overset{1024xy}{(\mathbf{x}-\mathbf{64})}=\overset{1}{\mathbf{0}} \qquad (22)$$

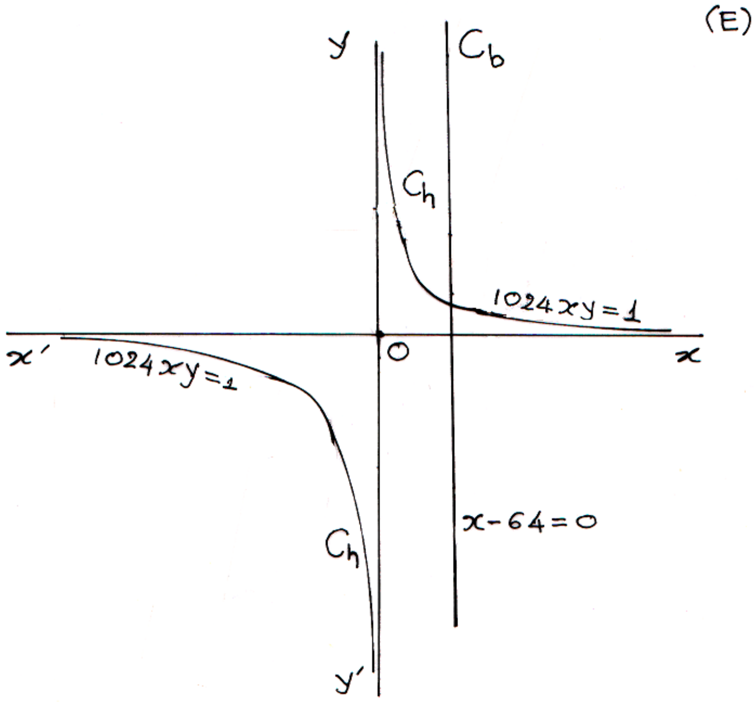
In the equation of degree (18) the base equation is:

$$\mathbf{x}-\mathbf{64}=\mathbf{0} \qquad (23)$$

and the hyperthetis equation:

$$\mathbf{1024xy = 1} \qquad (24)$$

Consequently, the graphical representation of the equation of degree (21), on a plane (E) in an orthogonal coordinate system **x-y**, is the line **x – 64 = 0** given by equation (23) and the hyperbola **1024xy = 1** given by equation (24), fig(7):



fig(7)

Example 4: Given the equation of degree (category $\overset{0}{\mathbf{C}}$),

$$\overset{0}{\mathbf{0}}+\overset{-1}{\mathbf{0}}+\overset{-2}{\mathbf{0}}+\overset{-3}{\mathbf{0}}+\overset{x^2}{\mathbf{0}}+\overset{y^2}{\mathbf{0}}+\overset{2}{\mathbf{x}}+\overset{3}{\mathbf{y}}-\left(\overset{1,5}{\mathbf{2}}\right)^{\overset{0}{2}}=\overset{0}{\mathbf{0}} \qquad (25)$$

represent it graphically on a plane (E) in an orthogonal coordinate system **x-y**.

Solution:

Equation (25) can be rearranged to give:

$$\overset{0}{\mathbf{0}}+\overset{-1}{\mathbf{0}}+\overset{-2}{\mathbf{0}}+\overset{-3}{\mathbf{0}}+\overset{x^2}{\mathbf{0}}+\overset{y^2}{\mathbf{0}}+\overset{2}{\mathbf{x}}+\overset{3}{\mathbf{y}}-\overset{\overset{0}{2}}{\left(\overset{1,5}{\mathbf{2}}\right)}$$

$$\overset{0}{\mathbf{0}}+\overset{-1}{\mathbf{0}}+\overset{-2}{\mathbf{0}}+\overset{-3}{\mathbf{0}}+\overset{x^2}{\mathbf{0}}+\overset{y^2}{\mathbf{0}}+\overset{2}{\mathbf{x}}+\overset{3}{\mathbf{y}}-\overset{(1,5+0)2}{\left(\mathbf{2}^2\right)}$$

$$\overset{0}{\mathbf{0}}+\overset{-1}{\mathbf{0}}+\overset{-2}{\mathbf{0}}+\overset{-3}{\mathbf{0}}+\overset{x^2}{\mathbf{0}}+\overset{y^2}{\mathbf{0}}+\overset{2}{\mathbf{x}}+\overset{3}{\mathbf{y}}-\overset{3}{\mathbf{4}}=$$

$$\overset{x^2+y^2-1}{(\mathbf{x}+\mathbf{y}-\mathbf{4})}=\overset{0}{\mathbf{0}} \qquad (26)$$

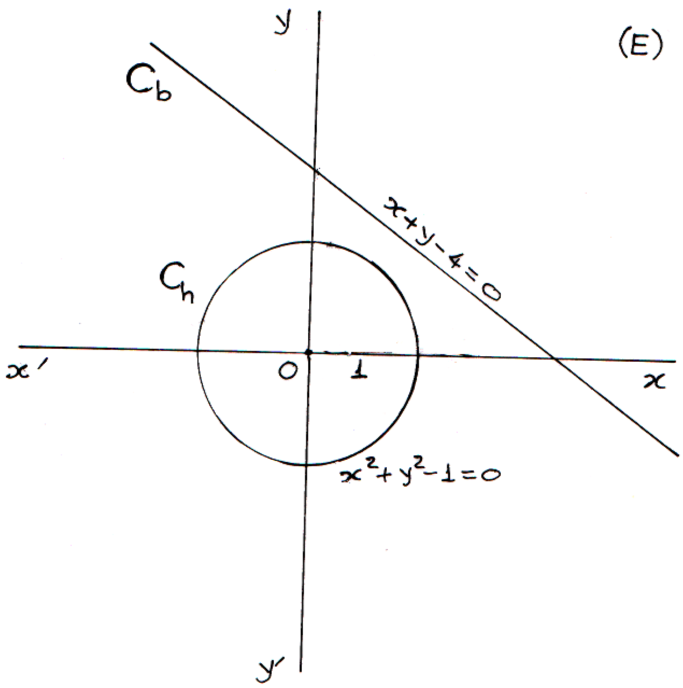
In the equation of degree (26) the base equation is:

$$\mathbf{x}+\mathbf{y}-\mathbf{4}=\mathbf{0} \qquad (27)$$

and the hyperthetis equation:

$$\mathbf{x}^2+\mathbf{y}^2-\mathbf{1}=\mathbf{0} \qquad (28)$$

Consequently, the graphical representation of the equation of degree (25), on a plane (E) in an orthogonal coordinate system **x-y**, is the line $\mathbf{x}+\mathbf{y}-\mathbf{4}=\mathbf{0}$ given by equation (27) and the circle $\mathbf{x}^2+\mathbf{y}^2-\mathbf{1}=\mathbf{0}$ given by equation (28), fig(8):



fig(8)

Example 5: Given the equation of degree (category $\overset{1}{\mathbf{C}}$),

$$\left(\overset{1}{\mathbf{x}}\right)^{\overset{2}{3}} + \left(\overset{2}{\mathbf{y}}\right)^{\overset{5}{3}} - \overset{x}{\mathbf{3}}\cdot \overset{3}{\boldsymbol{\alpha}}\cdot \overset{4}{\mathbf{x}}\cdot \overset{5}{\mathbf{y}} = \overset{1}{\mathbf{0}} \qquad (29)$$

represent it graphically on a plane (E) in an orthogonal coordinate system **x-y**.

Solution:

Equation (29) can be rearranged to give:

$$\left(\overset{1}{\mathbf{x}}\right)^{\overset{2}{3}} + \left(\overset{2}{\mathbf{y}}\right)^{\overset{5}{3}} - \overset{x}{\mathbf{3}}\cdot \overset{3}{\boldsymbol{\alpha}}\cdot \overset{4}{\mathbf{x}}\cdot \overset{5}{\mathbf{y}} =$$

$$\overset{(1\cdot 2)^3}{\left(\mathbf{x}^3\right)} + \overset{(2\cdot 5)^3}{\left(\mathbf{y}^3\right)} - \overset{x}{\mathbf{3}}\cdot \overset{3}{\boldsymbol{\alpha}}\cdot \overset{4}{\mathbf{x}}\cdot \overset{5}{\mathbf{y}} =$$

$$\overset{8}{\left(\mathbf{x}^3\right)} + \overset{1000}{\left(\mathbf{y}^3\right)} - \overset{x}{\mathbf{3}}\cdot \overset{3}{\boldsymbol{\alpha}}\cdot \overset{4}{\mathbf{x}}\cdot \overset{5}{\mathbf{y}} =$$

$$\overset{8\cdot 1000\cdot x\cdot 3\cdot 4\cdot 5}{\left(\mathbf{x}^3 + \mathbf{y}^3 - \mathbf{3}\boldsymbol{\alpha}\mathrm{x}\mathbf{y}\right)} =$$

$$\overset{480.000x}{\left(\mathbf{x}^3 + \mathbf{y}^3 - \mathbf{3}\boldsymbol{\alpha}\mathrm{x}\mathbf{y}\right)}, \text{ so:}$$

$$\overset{480.000x}{\left(\mathbf{x}^3 + \mathbf{y}^3 - \mathbf{3}\boldsymbol{\alpha}\mathrm{x}\mathbf{y}\right)} = \overset{1}{\mathbf{0}} \qquad (30)$$

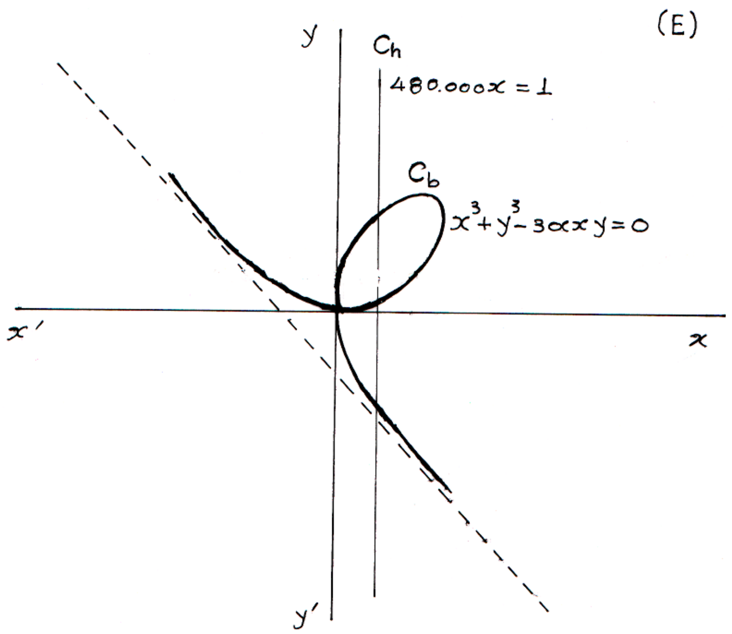
In the equation of degree (30) the base equation is:

$$\mathbf{x}^3 + \mathbf{y}^3 - \mathbf{3}\boldsymbol{\alpha}x\mathbf{y} = \mathbf{0} \qquad (31)$$

and the hyperthetis equation:

$$\mathbf{480.000x = 1} \qquad (32)$$

Consequently, the graphical representation of the equation of degree (29), on a plane (E) in an orthogonal coordinate system **x-y**, is the curve (Descartes leaf) $\mathbf{x}^3 + \mathbf{y}^3 - \mathbf{3}\boldsymbol{\alpha}\mathbf{xy} = \mathbf{0}$ given by equation (31) and the line **480.000x = 1** given by equation (32), fig(9):



fig(9)

Example 6: Given the equation of degree (category $\overset{\mathbf{0}}{\mathbf{C}}$),

$$\mathbf{x}^2\left(\overset{y}{\mathbf{2}}\right)^{\overset{3}{2}}+\mathbf{y}^2\left(\overset{x}{\mathbf{4}}\right)^{\overset{4}{2}}-\left(\overset{2}{\mathbf{4}}\right)^{\overset{5}{3}}=\overset{\mathbf{0}}{\mathbf{0}} \qquad (32.1)$$

represent it graphically on a plane (E) in an orthogonal coordinate system **x-y**.

Solution:

Equation (32.1) can be rearranged to give:

$$\mathbf{x}^2\left(\overset{y}{\mathbf{2}}\right)^{\overset{3}{2}}+\mathbf{y}^2\left(\overset{x}{\mathbf{4}}\right)^{\overset{4}{2}}-\left(\overset{2}{\mathbf{4}}\right)^{\overset{5}{3}}=$$

$$\mathbf{x}^2\left[\left(\overset{(y+3)2}{\mathbf{2}^2}\right)\right]+\mathbf{y}^2\left[\left(\overset{(x+4)2}{\mathbf{4}^2}\right)\right]-\left(\overset{(2+5)3}{\mathbf{4}^3}\right)=$$

$$\mathbf{x}^2\,\overset{\mathbf{2y+6}}{\mathbf{4}}+\mathbf{y}^2\,\overset{\mathbf{2x+8}}{\mathbf{16}}-\overset{\mathbf{21}}{\mathbf{64}}=$$

$$\left(\overset{\mathbf{0+(2y+6)+0+(2x+8)+21}}{\mathbf{x}^2\cdot\mathbf{4}+\mathbf{y}^2\cdot\mathbf{16}-\mathbf{64}}\right)=$$

$$\left(\overset{\mathbf{2x+2y+35}}{\mathbf{4x}^2+\mathbf{16y}^2-\mathbf{64}}\right), \text{ so:}$$

$$\left(\overset{\mathbf{2x+2y+35}}{\mathbf{4x}^2+\mathbf{16y}^2-\mathbf{64}}\right)=\overset{\mathbf{0}}{\mathbf{0}} \qquad (32.2)$$

In the equation of degree (32.2) the base equation is:

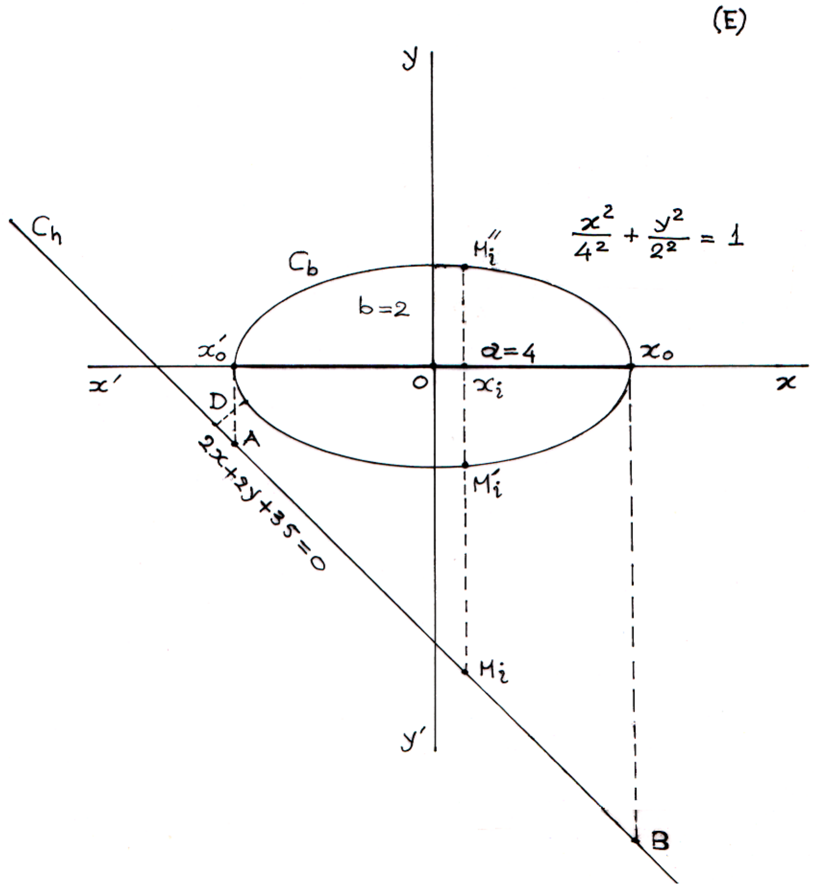
$$\mathbf{4x^2 + 16y^2 - 64 = 0 \Leftrightarrow \frac{x^2}{4^2} + \frac{y^2}{2^2} = 1} \qquad (32.3)$$

and the hyperthetis equation:

$$\mathbf{2x + 2y + 35 = 0} \qquad (32.4)$$

Consequently, the graphical representation of the equation of degree (32.1), on a plane (E) in an orthogonal coordinate system **x-y**, is the ellipse given by equation (32.3) and the line given by equation (32.4), fig(10):

Note: What is the graphical representation of the equation of degree (17) in the category $\overset{1}{\mathbf{C}}$? (Exercise for the reader)..



fig(10)

NOTEWORTHY REMARKS

1. The graph given by the base equation will be denoted as $\mathbf{C_b}$ and the graph given by the hyperthetis equation will be denoted by $\mathbf{C_h}$.

2. Solution of the simultaneous equations (base and hyperthetis) gives us the point(s) of intersection (if any exist) of the curves $\mathbf{C_b}$ and $\mathbf{C_h}$.

3. Consider for example fig(10) from example 6.

In this case we notice that for $\mathbf{x'_0} \le \mathbf{x} \le \mathbf{x_0}$, each $\mathbf{x} = \mathbf{x_i}$ corresponds to a point $\mathbf{M_i}$ of the line and two points $\mathbf{M'_i}$ and $\mathbf{M''_i}$ of the ellipse (non bijective). So, the segment AB of the line corresponds the whole of the circumference of the ellipse.

Conversely, in fig(6) from example 2, for $\mathbf{x_{-\infty}} \le \mathbf{x} \le \mathbf{x_{\infty}}$, each $\mathbf{x} = \mathbf{x_i}$ corresponds to the points $\mathbf{M_i}$ and $\mathbf{M'_i}$ on the two lines respectively (bijective). So, the whole of line $\mathbf{C_b}$ corresponds to the whole of the line $\mathbf{C_h}$.

4. To each equation of degree correspond two graphical representations, $\mathbf{C_b}$ and $\mathbf{C_h}$, either of category $\overset{\mathbf{1}}{\mathbf{C}}$ or $\overset{\mathbf{0}}{\mathbf{C}}$. Obviously $\mathbf{C_b}$ is the same for both categories whereas $\mathbf{C_h}$ is different.

5. The distance D between $\mathbf{C_b}$ and $\mathbf{C_h}$ is called the internal distance of the initial equation of degree. See fig(6&10).

6. Study of the two graphical representations $\mathbf{C_b}$ and $\mathbf{C_h}$ of an equation of degree solved for both categories $\overset{\mathbf{1}}{\mathbf{C}}$ and $\overset{\mathbf{0}}{\mathbf{C}}$ (full solution), constitutes the main field of study of Analytic Geometry of Degree.

The above analysis can be extended to three-dimensional space.
In this case the equations of degree would be of the form:

$$\mathbf{f_g}(\mathbf{x},\mathbf{y},\mathbf{z}) = \overset{\mathbf{1}}{\mathbf{0}}, \text{ (category } \overset{\mathbf{1}}{\mathbf{C}}\text{)}, \tag{33}$$

and

$$\mathbf{f_g}(\mathbf{x},\mathbf{y},\mathbf{z}) = \overset{\mathbf{0}}{\mathbf{0}}, \text{ (category } \overset{\mathbf{0}}{\mathbf{C}}\text{)}, \tag{34}$$

Obviously, the simultaneous equations arising from (33):

$$\left.\begin{aligned} \mathbf{f_b}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{0} \\ \mathbf{f_h}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{1} \end{aligned}\right\} \tag{35}$$

as the simultaneous equations arising from (34):

$$\left.\begin{array}{l}\mathbf{f_b}(\mathbf{x},\mathbf{y},\mathbf{z})=\mathbf{0}\\ \mathbf{f_h}(\mathbf{x},\mathbf{y},\mathbf{z})=\mathbf{0}\end{array}\right\} \qquad (36)$$

produce the graphs of the equations of degree (33) and (34) respectively.

Having said that, we conclude that the familiar Analytic Geometry is a partial case the Analytic Geometry of Degree.

Specifically, the equations of the familiar Analytic Geometry are equations of degree, in which all the numbers have hyperthetis 1 or 0 for categories $\overset{1}{\mathbf{C}}$ and $\overset{0}{\mathbf{C}}$ respectively. Consequently, in the familiar Analytic Geometry, rather than a pair of graphs $\mathbf{C_b}$ and $\mathbf{C_h}$, we have only a single graph, namely $\mathbf{C_b}$, that is the same in both categories $\overset{1}{\mathbf{C}}$ and $\overset{0}{\mathbf{C}}$.

Finally, as we can see, the field of study of Analytic Geometry of Degree is much broader than that of Analytic Geometry that we use to date.

Note: «Mathematics of Degree», see www.tsolkas.gr